Bethe–Salpeter Equation

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Abstract

We review the Bethe–Salpeter equation (BSE) in terms of a theory with a bilocal source $I(x, y)$. This summary covers sections 8.5 through 8.7 of Ref. [1]. Section 8.5 introduces the Bethe–Salpeter equation and its interaction kernel. In section 8.6 we apply this kernel to the NJL model and QED and section 8.7 investigates the stability criterion of the effective potential.

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1 Introduction

Our theory is completely described by the generating functional Z:

$$Z[J] = \int \mathcal{D}\varphi \; \mathrm{e}^{\mathrm{i}S[\varphi]+\varphi(x)J(x)}. \tag{1}$$

This functional $Z[J]$ contains the information off all transitions and interactions between some initial and final states. The expression in eq. (1) involves the calculation of a path integral, which is non-trivial. However, we could get to exactly the *same functional* without a path integral by using a different term in place of S,

$$Z[J] = \mathrm{e}^{\mathrm{i}\Gamma[\varphi_c]}, \tag{2}$$

which is called the effective action $\Gamma[\varphi]$, evaluated at some field configuration $\varphi_c(x)$. This field is called “classical field” and is defined as the solution of the equation[1]

$$\left.\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\right|_{\varphi=\varphi_c} = -J(x). \tag{3}$$

The reason why $\varphi_c(x)$ is called “classical” is that we do not need to calculate a path integral in order to get Z. If we had an expression for the effective action $\Gamma[\varphi]$, then – like in classical mechanics – we would just solve the equations of motion[2] and use the result in the exponential to get Z. As a final remark, due to $\Gamma = W - J\varphi$, the classical field $\varphi_c(x)$ is related to the quantum field $\varphi(x)$ via (see Appendix A.1 for the calculation),

$$\varphi_c(x) = \langle 0|\, \mathsf{T}\varphi(x)\, |0\rangle \tag{4}$$

We consider now a theory with a local source $J(x)$ *and* a bilocal source $I(x,y)$, which enters the theory in the following way:

$$S[\varphi] \to S[\varphi] + \varphi(x)J(x) + \varphi(x)\varphi(y)I(x,y) \tag{5}$$

Therefore our effective action will depend on two classical quantities,

$$\varphi_c(x) := \langle 0|\, \mathsf{T}\varphi(x)\, |0\rangle\,, \qquad G_c(x,y) := \mathrm{i}\,\langle 0|\, \mathsf{T}\varphi(x)\varphi(y)\, |0\rangle \tag{6}$$

where $\varphi_c(x)$ is the vacuum expectation value of the field in the presence of the sources J and G is the two-point propagator in the presence of J. If we set the sources to zero, G will be the usual propagator.

2 DSE and BSE – General Considerations (8.5)

2.1 Dyson–Schwinger Equation

The DSEs are named after Freeman Dyson [2] and Julian Schwinger [3, 4]. We already encountered the Dyson–Schwinger equation as the variational equation to determine G on the vacuum[3],

$$\frac{\delta\Gamma[\varphi_c, G]}{\delta G_{xy}} = 0 \tag{8.65}$$

[1] This classical field $\varphi_c(x)$ is a function of the source J. This is why the generating functional in eq. (2) still depends on the source J.

[2] This is exactly what happens in eq. (3). The derivative of any action with respect to a field yields the e.o.m. of said field.

[3] We use a shorthand notation $G_{xy} \equiv G(x,y)$, and $G^{-1}_{xy} \equiv [G(y,x)]^{-1}$ such that $G^{-1}_{xy}G_{yz} = \delta_{xz}$.

as well as an explicit expression for the effective action, called the loop-expansion:

$$\Gamma[\varphi_c, G] = \underbrace{S[\varphi_c]}_{\text{classical action}} + \frac{\mathrm{i}}{2}\operatorname{Tr}\operatorname{Ln} G^{-1} + \frac{\mathrm{i}}{2}\operatorname{Tr}\Big[\mathcal{D}^{-1}(\varphi_c)G\Big] + \underbrace{\Gamma_2[\varphi_c, G]}_{\text{2PI vacuum graphs with } L\geq 2} + C \tag{8.24}$$

We use the expression for Γ in eq. (8.24) to calculate the derivative in eq. (8.65),

$$\frac{\delta\Gamma[\varphi_c, G]}{\delta G_{xy}} = 0 + \frac{\mathrm{i}}{2}\underbrace{G_{ab}\Big(- G^{-1}_{bc}\tfrac{1}{2}(\delta_{cx}\delta_{dy} + \delta_{cy}\delta_{dx})G^{-1}_{da}\Big)}_{-G^{-1}_{yx}} + \frac{\mathrm{i}}{2}\underbrace{\mathcal{D}^{-1}_{ab}\delta_{bx}\delta_{ay}}_{\mathcal{D}^{-1}_{yx}} + \frac{\delta\Gamma_2[\varphi_c, G]}{\delta G_{xy}} \tag{7}$$

Here we used[4] $\frac{\partial G_{ab}}{\partial G_{xy}} = \frac{1}{2}(\delta_{ax}\delta_{by} + \delta_{ay}\delta_{bx})$, $G(x, y) = G(y, x)$ and $\mathrm{d}M^{-1} = -M^{-1}\mathrm{d}MM^{-1}$. With this, the DSE for G reads,

$$G^{-1}_{yx} = \mathcal{D}^{-1}_{yx} - \Sigma_{yx}(\varphi_c, G), \qquad \text{with } \Sigma_{yx}(\varphi_c, G) = 2\mathrm{i}\frac{\delta\Gamma_2[\varphi_c, G]}{\delta G_{xy}} \tag{8.66-67}$$

where we defined the self-energy operator Σ.

2.2 Bethe–Salpeter Equation

The BSE is named after Hans Bethe and Edwin Salpeter [5]. It describes bound states in a quantum field theory. The so-called Bethe–Salpeter wave function was shown to contain the same information as the S-matrix, thus it contains all the necessary information to describe the dynamics of bound states. The interesting object here is the four-point Green's function, $G^{(4)}(x, y, z, w)$, which we will insted view as a two-point function corresponding to two derivatives with respect to the bilocal source:

$$\begin{aligned} G^{(2)}_c(J; x_1y_1, x_2y_2) &= \frac{\delta^2 \mathrm{i}W[J, I]}{\mathrm{i}\delta I(x_1, y_1)\mathrm{i}\delta I(x_2, y_2)}\bigg|_{I=0} \\ &= \frac{\delta}{\mathrm{i}\delta I(x_1, y_1)}\left[\frac{1}{\mathcal{Z}}\frac{\delta\mathcal{Z}}{\mathrm{i}\delta I(x_2, y_2)}\right]\bigg|_{I=0} \\ &= \left[\frac{1}{\mathcal{Z}}\frac{\delta^2\mathcal{Z}}{\mathrm{i}\delta I(x_1, y_1)\mathrm{i}\delta I(x_2, y_2)} - \frac{1}{\mathcal{Z}^2}\frac{\delta\mathcal{Z}}{\mathrm{i}\delta I(x_1, y_1)}\frac{\delta\mathcal{Z}}{\mathrm{i}\delta I(x_2, y_2)}\right]\bigg|_{I=0} \\ &= \frac{1}{\mathcal{Z}[J]}G^{(2)}(J; x_1y_1, x_2y_2) - \frac{1}{\mathcal{Z}^2[J]}G^{(1)}(J; x_1y_1)G^{(1)}(J; x_2y_2) \end{aligned} \tag{8.10}$$

We define the BS operator as the second derivative of the effective action with respect to G:

$$\mathrm{i}\frac{\delta^2\Gamma}{\delta G_{x_1y_1}\delta G_{x_2y_2}} \tag{8}$$

[4] For scalar fields we have to introduce this kind of symmetrization. Later when we talk about fermions, only the first two Kronecker deltas would remain.

We calculate this expression using the loop expansion of Γ, eq. (8.24),

$$\begin{aligned}
\frac{\delta^2\Gamma}{\delta G_{ij}\delta G_{kl}} &= \frac{\mathrm{i}}{2}\frac{\delta}{\delta G_{ij}}\Big[-\underbrace{G_{ab}G^{-1}_{bc}}_{\delta_{ac}}\tfrac{1}{2}(\delta_{ck}\delta_{dl}+\delta_{cl}\delta_{dk})G^{-1}_{da}\Big]+\frac{\delta^2\Gamma_2}{\delta G_{ij}\delta G_{kl}}\\
&= -\frac{\mathrm{i}}{2}\frac{\delta}{\delta G_{ij}}\Big[G^{-1}_{lk}\Big]+\frac{\delta^2\Gamma_2}{\delta G_{ij}\delta G_{kl}}\\
&= \frac{\mathrm{i}}{2}\Big[G^{-1}_{la}\tfrac{1}{2}(\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi})G^{-1}_{bk}\Big]+\frac{\delta^2\Gamma_2}{\delta G_{ij}\delta G_{kl}}\\
&= \frac{\mathrm{i}}{4}\Big[G^{-1}_{li}G^{-1}_{jk}+G^{-1}_{lj}G^{-1}_{ik}\Big]+\frac{\delta^2\Gamma_2}{\delta G_{ij}\delta G_{kl}} \qquad (9)
\end{aligned}$$

If we replace $(ijkl)$ with $(x_1y_1x_2y_2)$, we get

$$\mathrm{i}\frac{\delta^2\Gamma}{\delta G_{x_1y_1}\delta G_{x_2y_2}} = -\frac{1}{4}\Big[G^{-1}_{y_2x_1}G^{-1}_{y_1x_2}+G^{-1}_{y_2y_1}G^{-1}_{x_1x_2}\Big]+\mathrm{i}\frac{\delta^2\Gamma_2}{\delta G_{x_1y_1}\delta G_{x_2y_2}} \qquad (8.70)$$

In order to get the BS equation, we apply the BS operator onto $G^{(2)}_c$. Remember that

$$\frac{\delta^2\mathrm{i}\Gamma}{\delta G_{ab}\delta G_{xy}}\underbrace{\frac{\delta^2\mathrm{i}W[J,I]}{\mathrm{i}\delta I_{xy}\mathrm{i}\delta I_{cd}}}_{G^{(2)}_c{}_{xy,cd}} = \frac{1}{2}\Big(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\Big) \qquad (8.18)$$

We use the expression eq. (8.70) in eq. (8.18):

$$-\frac{1}{4}\Big[G^{-1}_{da}G^{-1}_{bc}+G^{-1}_{db}G^{-1}_{ac}\Big]G^{(2)}_c{}_{cd,ef}+\mathrm{i}\frac{\delta^2\Gamma_2}{\delta G_{ab}\delta G_{cd}}G^{(2)}_c{}_{cd,ef} = \frac{1}{2}\Big(\delta_{ae}\delta_{bf}+\delta_{af}\delta_{be}\Big) \qquad (10)$$

Multiply with $(G_{gb}G_{ah}+G_{ga}G_{bh})$. The first and third terms become:

$$(G_{gb}G_{ah}+G_{ga}G_{bh})\Big[G^{-1}_{da}G^{-1}_{bc}+G^{-1}_{db}G^{-1}_{ac}\Big]G^{(2)}_c{}_{cd,ef} = 4G^{(2)}_c{}_{gh,ef} \qquad (11)$$

$$(G_{gb}G_{ah}+G_{ga}G_{bh})\Big(\delta_{ae}\delta_{bf}+\delta_{af}\delta_{be}\Big) = 2(G_{gf}G_{eh}+G_{ge}G_{fh}) \qquad (12)$$

Therefore we can write the BSE as

$$-G^{(2)}_c{}_{gh,ef}+\mathrm{i}(G_{gb}G_{ah}+G_{ga}G_{bh})\frac{\delta^2\Gamma_2}{\delta G_{ab}\delta G_{cd}}G^{(2)}_c{}_{cd,ef} = (G_{gf}G_{eh}+G_{ge}G_{fh}) \qquad (8.71)$$

Finally, we introduce two abbreviations,

$$G^{(2)}_0(J;x_1y_1,x_2y_2) \equiv -\Big[G(x_1,y_2)G(x_2,y_1)+G(x_1,x_2)G(y_2,y_1)\Big] \qquad (8.73)$$

$$K(J;x_1y_1,x_2y_2) \equiv -\mathrm{i}\frac{\delta^2\Gamma_2}{\delta G(x_1,y_1)\delta G(x_2,y_2)} \qquad (8.74)$$

to write the BS equation in the usual way:

$$G^{(2)}_c{}_{x_1y_1,x_2y_2} = G^{(2)}_0{}_{x_1y_1,x_2y_2}+G^{(2)}_0{}_{x_1y_1,xy}K_{xy,x'y'}G^{(2)}_c{}_{x'y',x_2y_2} \qquad (8.72)$$

This is an iterative equation: one can use the whole expression in the last term again and again.

2.2.1 Going to Momentum Space

Before we continue, we make three assumptions:

- We set the sources to zero: $J(x) = 0$ and $I(x, y) = 0$.
- We assume that the fields at position x_i have mass m_x and those at position y_i have a different mass m_y. This means we have to change eq. (8.73) to

  $$G_0^{(2)}(J; x_1y_1, x_2y_2) = -G(x_1, x_2)G(y_2, y_1) \tag{8.75}$$

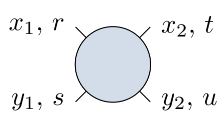
  because propagation through spacetime cannot change the mass of a particle.
- By assuming translational invariance, we do not need four coordinates (x_1, y_1, x_2, y_2) to describe our Green's functions, since we can freely move around the origin. We only need three independent coordinates and we choose them to be

  $$z_1 = x_1 - y_1, \quad z_2 = y_2 - x_2, \quad z_3 = \eta_x(x_1 - x_2) + \eta_y(y_1 - y_2). \tag{8.76}$$

  The constants η_i are determined by the particles' masses:

  $$\eta_x = \frac{m_x}{m_x + m_y}, \quad \eta_y = \frac{m_y}{m_x + m_y} \quad \rightarrow \quad \eta_x + \eta_y = 1 \tag{13}$$

In order to change to momentum space, we introduce three conjugate momenta: (z_1, p), (z_2, q) and (z_3, P). From now on we will also explicitly indicate the (spinor, flavor, color combined) components (r, s, t, u) of the Green's functions:



$$G^{(2)}_{c;rs,tu}(x_1y_1, x_2y_2) = \int_{p^\mu}\int_{q^\mu}\int_{P^\mu} \mathrm{e}^{-\mathrm{i}(z_1p+z_2q+z_3P)} G^{(2)}_{c;rs,tu}(p, q; P) \tag{8.77}$$

$$K_{rs,tu}(x_1y_1, x_2y_2) = \int_{p^\mu}\int_{q^\mu}\int_{P^\mu} \mathrm{e}^{-\mathrm{i}(z_1p+z_2q+z_3P)} K_{rs,tu}(p, q; P) \tag{8.78}$$

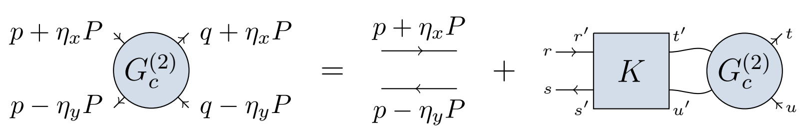
Now we can write the BSE in momentum space:

$$\begin{aligned} G^{(2)}_{c;rs,tu}(p, q; P) &= G^{(2)}_{0;rs,tu}(p, q; P) \\ &\quad + \int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} G^{(2)}_{0;rs,r's'}(p, \tilde{p}; P)K_{r's',t'u'}(\tilde{p}, \tilde{q}; P)G^{(2)}_{c;t'u',tu}(\tilde{q}, q; P) \end{aligned} \tag{8.79}$$

where (see appendix A.2 for a derivation)

$$G^{(2)}_{0;rs,tu}(p, q; P) = -(2\pi)^4\delta^{(4)}(p - q)G_{rt}(p + \eta_xP)G_{us}(p - \eta_yP) \tag{8.80}$$

Using diagrams, eq. (8.79) looks like



(14)

2.2.2 Homogenous BSE

The homogenous Bethe–Salpeter equation describes bound states. We define these bound states via a ket vector $|P_b, r\rangle$. These bound states have four-momentum P_b^μ and we assume that there might be $i = 1, .., k$ degenerate bound states in our theory. They are normalized according to

$$\langle P_b, i|P_b', i'\rangle = (2\pi)^3\, 2P_b^0\, \delta^{(3)}(\vec{P}_b - \vec{P}_b') \tag{8.83}$$

Now we can define the Bethe–Salpeter wave function χ:

$$\chi_{i;rs}(x, y; P_b) = \langle 0|\mathsf{T}\varphi_r(x)\varphi_s(y)|P_b, i\rangle \tag{8.81}$$

This equation implies that the BS wave function plays the role of a form factor for a certain bound state b. Its conjugate is:

$$\bar{\chi}_{i;sr}(y, x; P_b) = \langle P_b, i|\mathsf{T}\varphi_r(x)\varphi_s(y)|0\rangle = \Big[\, \langle 0|\bar{\mathsf{T}}\varphi_r(x)\varphi_s(y)|P_b, i\rangle \,\Big]^* \tag{8.82}$$

Due to translational invariance, χ only depends on one independent variable, for instance $z = x - y$ (see appendix A.3 for details):

$$\chi_{i;rs}(x, y, P_b) = \mathrm{e}^{-\mathrm{i}XP_b}\chi_{i;rs}(z, P_b), \qquad \bar{\chi}_{i;rs}(x, y, P_b) = \mathrm{e}^{\mathrm{i}XP_b}\bar{\chi}_{i;rs}(z, P_b) \tag{8.84}$$

Therefore the momentum-space BS wave functions only depend on one momentum:

$$\chi_{i;rs}(p, P_b) = \int_{z^\mu} \mathrm{e}^{\mathrm{i}zp}\, \chi_{i;rs}(z, P_b), \qquad \bar{\chi}_{i;rs}(p, P_b) = \int_{z^\mu} \mathrm{e}^{\mathrm{i}zp}\, \bar{\chi}_{i;rs}(z, P_b) \tag{8.88}$$

One can show that the Green's function $G_c^{(2)}(p, q, P)$ has a diverging pole term when the momentum that travels through the Green's function is equal to the momentum of a bound state, i.e. $P \to \pm P_b$ (see A.4 for the details).

$$G^{(2)}_{c;rs,tu}(p, q; P)\bigg|_{\text{poles}} = \frac{\mathrm{i}\sum_{i=1}^k \chi_{i;rs}(p, \vec{P}_b)\bar{\chi}_{i;tu}(q, \vec{P}_b)}{P^2 - P_b^2 + \mathrm{i}\epsilon} \tag{8.87}$$

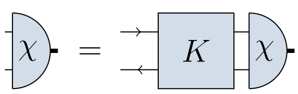
We will use this result in eq. (8.79) to get the homogenous Bethe–Salpeter equation. In the limit of a bound state, the first term with only propagators is finite, whereas the other terms diverge. Therefore only the second term on the right-hand side remains.

$$\chi_{i;rs}(p, P_b)\bar{\chi}_{i;tu}(q, P_b) = \int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} G^{(2)}_{0;rs,r's'}(p, \tilde{p}; P)K_{r's',t'u'}(\tilde{p}, \tilde{q}; P)\chi_{i;t'u'}(\tilde{q}, P_b)\bar{\chi}_{i;tu}(q, P_b) \tag{15}$$

Since we do not integrate over q^μ, we can multiply with the inverse of $\bar{\chi}$ on both sides and eliminate it.

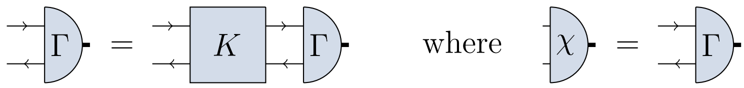
$$\chi_{i;rs}(p, P_b) = \int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} G^{(2)}_{0;rs,r's'}(p, \tilde{p}; P)K_{r's',t'u'}(\tilde{p}, \tilde{q}; P)\chi_{i;t'u'}(\tilde{q}, P_b) \tag{16}$$

This equation is equivalent to these diagrams:



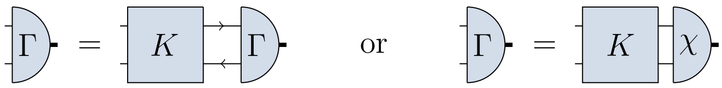
(17)

We can get a different shape of this equation by defining the Bethe–Salpeter amplitude Γ as $\chi_{rs} = G_{rr'}\Gamma_{r's'}G_{s's}$. To change from the BS wave function to the BS amplitude, we simply remove the two propagators from its legs. Now we can draw eq. (17) in a new way:



(18)

Now it is apparent that we can remove the left-most propagators on both sides of the equation, which leaves us with the diagrams



(19)

and the equation looks like this:

$$-G^{-1}_{rr'}(p+\eta_x P)\chi_{i;r's'}(p,P_b)G^{-1}_{s's}(p-\eta_y P) = \int_{q^\mu} K_{rs,tu}(p,q,P_b)\chi_{i;tu}(q,P_b) \tag{8.89}$$

To conclude this chapter, we will now discuss an even shorter way to write down the BSE, which is called the "operator form". The inhomogenous BSE, eq. (8.79), can be written like

$$G^{(2)}_c = G^{(2)}_0 + G^{(2)}_0 K G^{(2)}_c \tag{8.90}$$

This is a recursive equation, so we might as well write

$$G^{(2)}_c = G^{(2)}_0 + G^{(2)}_0 K G^{(2)}_0 + G^{(2)}_0 K G^{(2)}_0 K G^{(2)}_0 + \cdots = G^{(2)}_0 \sum_{i=0}^{\infty}\left(K G^{(2)}_0\right)^i = \frac{G^{(2)}_0}{1-KG^{(2)}_0} \tag{20}$$

This result can be rewritten to

$$G^{(2)}_c = \left((G^{(2)}_0)^{-1} - K\right)^{-1} \tag{8.91}$$

And for the homogenous BSE, the operator form looks like (compare eq. (16))

$$(G^{(2)}_0)^{-1}\chi = K\chi \tag{8.92}$$

2.2.3 Normalization of the BS Wave Function

Since the homogenous BSE allows a rescaling $\chi \to const.\,\chi$, we need to impose some kind of normalization condition on the BS wavefunction. A trick to get this normalization condition is to introduce a parameter η to the scattering kernel[5] K of the inhomogenous BSE. In the inhomogenous BSE, we do not have the freedom of rescaling, therefore it should help us find a normalization condition:

$$G^{(2)}_c(\eta) = G^{(2)}_0 + G^{(2)}_0 \eta K G^{(2)}_c(\eta) \tag{8.93}$$

We differentiata both sides with respect to η:

$$\frac{\partial G^{(2)}_c(\eta)}{\partial\eta} = G^{(2)}_0 K G^{(2)}_c(\eta) + G^{(2)}_0 \eta K \frac{\partial G^{(2)}_c(\eta)}{\partial\eta} \tag{21}$$

[5] If the kernel depends linearly on a coupling constant g, then this coupling can take the role of η.

We subtract the second term on the right to the left-hand side of the equation:

$$\left(1 - G_0^{(2)}\eta K\right)\frac{\partial G_c^{(2)}(\eta)}{\partial\eta} = G_0^{(2)} K G_c^{(2)}(\eta) \tag{22}$$

and multiply with the inverse of the brackets from the left:

$$\frac{\partial G_c^{(2)}(\eta)}{\partial\eta} = \left(1 - G_0^{(2)}\eta K\right)^{-1} G_0^{(2)} K G_c^{(2)}(\eta) \tag{23}$$

A quick calculation shows that we can manipulate eq. (8.93) in the following way,

$$G_c^{(2)}(\eta) = G_0^{(2)} + G_0^{(2)}\eta K G_c^{(2)}(\eta) \tag{24a}$$

$$1 = G_0^{(2)}[G_c^{(2)}(\eta)]^{-1} + G_0^{(2)}\eta K \tag{24b}$$

$$1 - G_0^{(2)}\eta K = G_0^{(2)}[G_c^{(2)}(\eta)]^{-1} \tag{24c}$$

$$\left(1 - G_0^{(2)}\eta K\right)^{-1} = G_c^{(2)}(\eta)[G_0^{(2)}]^{-1} \tag{24d}$$

thereby giving us a way to replace the terms in brackets.

$$\frac{\partial G_c^{(2)}(\eta)}{\partial\eta} = G_c^{(2)}(\eta)[G_0^{(2)}]^{-1}G_0^{(2)} K G_c^{(2)}(\eta) = G_c^{(2)}(\eta) K G_c^{(2)}(\eta) \tag{8.95}$$

In terms of the BS wave function, the normalization condition can be written in a different way. Let us first evaluate the left-hand side of eq. (8.95),

$$\begin{aligned}\frac{\partial G_c^{(2)}(\eta)}{\partial\eta} &= \mathrm{i}\frac{\partial}{\partial\eta}\frac{\chi\bar\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon}\\ &= \mathrm{i}\frac{\partial\chi}{\partial\eta}\frac{\bar\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon} + \mathrm{i}\frac{\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon}\frac{\partial\bar\chi}{\partial\eta} - \mathrm{i}\frac{\chi\bar\chi}{(K^2 - P_b^2 + \mathrm{i}\epsilon)^2}\left(-\frac{\partial P_b^2}{\partial\eta}\right)\end{aligned} \tag{25}$$

This is because if a parameter in the scattering kernel changes (i.e. η changes), both the wavefunction χ as well as the eigenvalue P_b^2 will change and thus depend on η. We can now write eq. (8.95) as

$$\begin{aligned}\mathrm{i}\frac{\partial\chi}{\partial\eta}\frac{\bar\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon} + \mathrm{i}\frac{\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon}\frac{\partial\bar\chi}{\partial\eta} - \mathrm{i}\frac{\chi\bar\chi}{(K^2 - P_b^2 + \mathrm{i}\epsilon)^2}\left(-\frac{\partial P_b^2}{\partial\eta}\right)&\\ = -\frac{\chi\bar\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon}K\frac{\chi\bar\chi}{K^2 - P_b^2 + \mathrm{i}\epsilon}&\end{aligned} \tag{26}$$

We multiply with $(K^2 - P_b^2 + \mathrm{i}\epsilon)^2$ and set K^2 equal to P_b^2 (i.e. only look at the double pole terms),

$$-\mathrm{i}\chi\bar\chi\left(-\frac{\partial P_b^2}{\partial\eta}\right) = -\chi\bar\chi K\chi\bar\chi \tag{27}$$

We can get rid of the out-most χ and $\bar\chi$ by multiplying with their inverses,

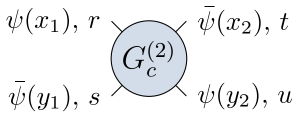
$$\mathrm{i}\frac{\partial P_b^2}{\partial\eta} = -\bar\chi K\chi \tag{28}$$

Finally we replace $K\chi$ on the right-hand side using eq. (8.92) (remember to introduce the parameter η to this equation!) to get the normalization condition,

$$\eta\frac{\partial P_b^2}{\partial\eta} = \mathrm{i}\bar\chi(G_0^{(2)})^{-1}\chi. \tag{8.96}$$

2.2.4 Fermions

The Bethe–Salpeter formalism can be easily adapted for fermions, e.g. quarks that build up hadrons. Now, $G_c^{(2)}$ is defined as (note the order of the arguments x, y)



$$G^{(2)}_{c;rs,ut}(x_1 y_1, y_2 x_2) = \langle 0|\mathsf{T}\psi_r(x_1)\bar{\psi}_s(y_1)\psi_u(y_2)\bar{\psi}_t(x_2)|0\rangle \tag{8.98}$$

The inhomogenous BSE is now (compare to eq. (8.79) and notice the indices)

$$\begin{aligned} G^{(2)}_{c;rs,ut}(p,q;P) &= G^{(2)}_{0;rs,ut}(p,q;P) \\ &\quad + \int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} G^{(2)}_{0;rs,s'r'}(p,\tilde{p};P) K_{s'r',t'u'}(\tilde{p},\tilde{q};P) G^{(2)}_{c;t'u',ut}(\tilde{q},q;P) \end{aligned} \tag{8.99}$$

where G_0 is given by (compare to eq. (8.80), no minus sign here)

$$G^{(2)}_{0;rs,ut}(p,q;P) = (2\pi)^4 \delta^{(4)}(p-q) G_{rt}(p+\eta_x P) G_{us}(p-\eta_y P) \tag{8.100}$$

The BS kernel K is (compare to eq. (8.74))

$$K_{sr,tu}(y_1 x_1, x_2 y_2) = -\mathrm{i}\frac{\delta^2 \Gamma_2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)} \tag{8.101}$$

The BS wave functions for fermions read (compare to eq. (8.81))

$$\chi_{i;rs}(x,y;P_b) = \langle 0|\mathsf{T}\psi_r(x)\bar{\psi}_s(y)|P_b,i\rangle\,, \quad \bar{\chi}_{i;rs}(y,x;P_b) = \langle P_b,i|\mathsf{T}\psi_s(x)\bar{\psi}_r(y)|0\rangle \tag{8.102}$$

Due to translational invariance, their momentum-space representation can be written as (compare to eqs. (8.84) and (8.88))

$$\begin{aligned} \chi_{i;rs}(x,y;P_b) &= \mathrm{e}^{-\mathrm{i}XP_b}\chi_{i;rs}(z;P_b) = \mathrm{e}^{-\mathrm{i}XP_b}\int_{p^\mu} \mathrm{e}^{-\mathrm{i}pz}\chi_{i;rs}(p,P_b) \\ \bar{\chi}_{i;rs}(y,x;P_b) &= \mathrm{e}^{\mathrm{i}XP_b}\bar{\chi}_{i;rs}(z;P_b) = \mathrm{e}^{\mathrm{i}XP_b}\int_{p^\mu} \mathrm{e}^{\mathrm{i}pz}\bar{\chi}_{i;rs}(p,P_b) \end{aligned} \tag{8.103}$$

with the center-of-mass coordinate $X = \frac{m_x x + m_y y}{m_x + m_y}$. The Green's function has a bound state pole like (compare to eq. (8.87))

$$G^{(2)}_{c;rs,ut}(p,q;P)\Big|_{\text{poles}} = \mathrm{i}\frac{\sum_{i=1}^{k}\chi_{i;rs}(p,\vec{P}_b)\bar{\chi}_{i;tu}(q,\vec{P}_b)}{P^2 - P_b^2 + \mathrm{i}\epsilon} \tag{8.104}$$

and the homogenous BSE looks like (compare to eq. (8.89))

$$G^{-1}_{rr'}(p+\eta_x P)\chi_{i;r's'}(p,P_b)G^{-1}_{s's}(p-\eta_y P) = \int_{q^\mu} K_{sr,tu}(p,q,P_b)\chi_{i;tu}(q,P_b) \tag{8.105}$$

The normalization condition is the same as we wrote down previously in eq. (8.96)

$$\eta\frac{\partial P_b^2}{\partial \eta} = \mathrm{i}\bar{\chi}(G_0^{(2)})^{-1}\chi \tag{$\to$ 8.96}$$

3 BSE for the NJL model and QED (8.6)

We remember that the interaction kernel of the Bethe–Salpeter equation for fermions was given in eq. (8.101),

$$K_{sr,tu}(y_1x_1, x_2y_2) = -\mathrm{i}\frac{\delta^2\Gamma_2}{\delta G_{sr}(y_1, x_1)\delta G_{tu}(x_2, y_2)} \qquad (\to 8.101)$$

We will now consider two concrete examples for Γ_2.

3.1 Nambu–Jona-Lasinio Model

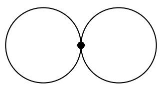
The Nambu–Jona-Lasinio (NJL) model [6, 7] introduces a four-fermion interaction. The Lagrangian for N_f equal-mass fermion flavors reads,

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m_0)\psi + g\sum_{a=0}^{N_f^2-1}\left\{\left(\bar{\psi}\mathrm{t}^a\psi\right)^2 + \left(\bar{\psi}\mathrm{t}^a\mathrm{i}\gamma^5\psi\right)^2\right\} \tag{8.46}$$

and we already calculated Γ_2 to be

$$\begin{aligned}\Gamma_2^{(\mathrm{NJL})} = -g\sum_{a=0}^{N_f^2-1}\int_{x^\mu}\Big\{&\Big(\mathrm{Tr}[\mathrm{t}^aG(x,x)]\Big)^2 + \Big(\mathrm{Tr}\big[\mathrm{t}^a\mathrm{i}\gamma^5G(x,x)\big]\Big)^2\\ &- \mathrm{Tr}[\mathrm{t}^aG(x,x)\mathrm{t}^aG(x,x)] - \mathrm{Tr}\big[\mathrm{t}^a\mathrm{i}\gamma^5G(x,x)\mathrm{t}^a\mathrm{i}\gamma^5G(x,x)\big]\Big\}\end{aligned} \tag{8.48}$$

Here we set $\psi_c = 0$ and used a two-loop approximation ($L = 2$). The terms in eq. (8.48) correspond this kind of diagram,



(29)

where the black dot represents either a coupling with or without a γ^5 matrix. We will now differentiate every term by itself:

$$K^{(1.)}_{sr,tu} = \mathrm{i}g\int_{x^\mu}\frac{\delta^2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)}\Big(\mathrm{t}^a_{AB}G_{BA}(x,x)\Big)^2 \tag{30a}$$

$$= \mathrm{i}g\int_{x^\mu}\frac{\delta}{\delta G_{sr}(y_1,x_1)}2\Big(\mathrm{t}^a_{CD}G_{DC}(x,x)\Big)\mathrm{t}^a_{AB}\delta^{Bt}_{Au}\delta^{(4)}(x-x_2)\delta^{(4)}(x-y_2) \tag{30b}$$

$$= 2\mathrm{i}g\int_{x^\mu}\mathrm{t}^a_{CD}\delta^{Ds}_{Cr}\delta^{(4)}(x-x_1)\delta^{(4)}(x-y_1)\mathrm{t}^a_{tu}\delta^{(4)}(x-x_2)\delta^{(4)}(x-y_2) \tag{30c}$$

$$= 2\mathrm{i}g\,\mathrm{t}^a_{rs}\mathrm{t}^a_{tu}\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_1-y_2) \tag{30d}$$

or in momentum space:

$$K^{(1.)}_{sr,tu}(p,q;P) = 2\mathrm{i}g\,\mathrm{t}^a_{rs}\mathrm{t}^a_{tu} \tag{31}$$

The second term in eq. (8.48) works very similar to the first term (note that the indices (r, s, A, B) are collective indices for *flavor and spinor* indices!):

$$K^{(2.)}_{sr,tu} = \mathrm{i}g\int_{x^\mu}\frac{\delta^2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)}\Big([\mathrm{t}^a\mathrm{i}\gamma^5]_{AB}G_{BA}(x,x)\Big)^2 \tag{32a}$$

$$= 2\mathrm{i}g\,[\mathrm{t}^a\mathrm{i}\gamma^5]_{rs}[\mathrm{t}^a\mathrm{i}\gamma^5]_{tu}\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_1-y_2) \tag{32b}$$

or in momentum space:

$$K^{(2.)}_{sr,tu}(p,q;P) = 2\mathrm{i}g\,[\mathsf{t}^a\mathrm{i}\gamma^5]_{rs}[\mathsf{t}^a\mathrm{i}\gamma^5]_{tu} \tag{33}$$

The third term in eq. (8.48) works slightly differently to the first and second terms:

$$\begin{aligned}
K^{(3.)}_{sr,tu} &= \mathrm{i}g\int_{x^\mu}\frac{\delta^2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)}\Big(\mathsf{t}^a_{AB}G_{BC}(x,x)\mathsf{t}^a_{CD}G_{DA}(x,x)\Big) && \text{(34a)}\\
&= \mathrm{i}g\int_{x^\mu}\frac{\delta}{\delta G_{sr}(y_1,x_1)}\Big(\mathsf{t}^a_{AB}\delta^{Bt}_{Cu}\mathsf{t}^a_{CD}G_{DA}(x,x)+\mathsf{t}^a_{AB}G_{BC}(x,x)\mathsf{t}^a_{CD}\delta^{Dt}_{Au}\Big)\\
&\qquad \delta^{(4)}(x-x_2)\delta^{(4)}(x-y_2) && \text{(34b)}\\
&= \mathrm{i}g\int_{x^\mu}\Big(\mathsf{t}^a_{At}\mathsf{t}^a_{uD}\delta^{Ds}_{Ar}+\mathsf{t}^a_{uB}\delta^{Bs}_{Cr}\mathsf{t}^a_{Ct}\Big)\delta^{(4)}(x-x_2)\delta^{(4)}(x-y_2)\\
&\qquad \delta^{(4)}(x-x_1)\delta^{(4)}(x-y_1) && \text{(34c)}\\
&= \mathrm{i}g\Big(\mathsf{t}^a_{rt}\mathsf{t}^a_{us}+\mathsf{t}^a_{us}\mathsf{t}^a_{rt}\Big)\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_1-y_2) && \text{(34d)}\\
&= 2\mathrm{i}g\,\mathsf{t}^a_{rt}\mathsf{t}^a_{us}\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_1-y_2) && \text{(34e)}
\end{aligned}$$

or in momentum space:

$$K^{(3.)}_{sr,tu}(p,q;P) = 2\mathrm{i}g\,\mathsf{t}^a_{rt}\mathsf{t}^a_{us} \tag{35}$$

The fourth and last term works again similarly to the third one.

$$\begin{aligned}
K^{(3.)}_{sr,tu} &= \mathrm{i}g\int_{x^\mu}\frac{\delta^2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)}\Big([\mathsf{t}^a\mathrm{i}\gamma^5]_{AB}G_{BC}(x,x)[\mathsf{t}^a\mathrm{i}\gamma^5]_{CD}G_{DA}(x,x)\Big) && \text{(36a)}\\
&= 2\mathrm{i}g\,[\mathsf{t}^a\mathrm{i}\gamma^5]_{rt}[\mathsf{t}^a\mathrm{i}\gamma^5]_{us}\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_1-y_2) && \text{(36b)}
\end{aligned}$$

or in momentum space:

$$K^{(4.)}_{sr,tu}(p,q;P) = 2\mathrm{i}g\,[\mathsf{t}^a\mathrm{i}\gamma^5]_{rt}[\mathsf{t}^a\mathrm{i}\gamma^5]_{us} \tag{37}$$

Finally, the NJL-model Bethe–Salpeter scattering kernel in momentum space reads,

$$K_{sr,tu}(p,q;P) = 2\mathrm{i}g\Big(\mathsf{t}^a_{rs}\mathsf{t}^a_{tu}+[\mathsf{t}^a\mathrm{i}\gamma^5]_{rs}[\mathsf{t}^a\mathrm{i}\gamma^5]_{tu}+\mathsf{t}^a_{rt}\mathsf{t}^a_{us}+[\mathsf{t}^a\mathrm{i}\gamma^5]_{rt}[\mathsf{t}^a\mathrm{i}\gamma^5]_{us}\Big) \tag{8.110}$$

We will meet the NJL model again in section 9.

3.2 Quantum Electrodynamics

The QED Lagrangian for equal-mass fermion flavors reads[6],

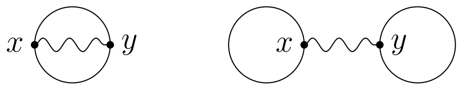
$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar\psi(\mathrm{i}\partial\!\!\!/-m_0)\psi-g\bar\psi\gamma^\mu\psi A_\mu-\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{8.56}$$

and we already calculated Γ_2 to be

$$\begin{aligned}
\Gamma_2^{(\mathrm{QED})} &= \frac{g^2}{2}\int_{x^\mu}\int_{y^\mu}\mathrm{Tr}\Big[G(x,y)\gamma^\mu G(y,x)\gamma^\nu\Big]D_{\mu\nu}(x-y)\\
&\quad -\frac{g^2}{2}\int_{x^\mu}\int_{y^\mu}\mathrm{Tr}\Big[G(x,x)\gamma^\mu\Big]D_{\mu\nu}(x-y)\,\mathrm{Tr}\Big[G(y,y)\gamma^\nu\Big] && \text{(8.59)}
\end{aligned}$$

[6]We redefine $\xi\to\xi^{-1}$ such that $\xi=0$ leads to the Landau gauge, as usual.

Here we used a two-loop approximation ($L = 2$) and set $\psi_c = 0$, $A_c^\mu = 0$, which leads to replacing the full photon propagator $\Delta_{\mu\nu}$ with the free photon propagator $D_{\mu\nu}$. Also, we assumed that there is no bilocal source for the photon field. The two terms in eq. (8.59) correspond to the diagrams

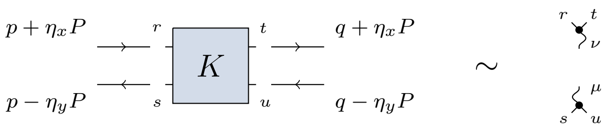


(38)

We will calculate the two terms in eq. (8.59) separately:

$$
\begin{aligned}
K^{(1.)}_{sr,tu} &= \frac{g^2}{2\mathrm{i}}\int_{x^\mu}\int_{y^\mu}\frac{\delta^2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)}\Big(G_{AB}(x,y)\gamma^\mu_{BC}G_{CD}(y,x)\gamma^\nu_{DA}\Big)D_{\mu\nu}(x-y) && \text{(39a)}\\
&= \frac{g^2}{2\mathrm{i}}\int_{x^\mu}\int_{y^\mu}\frac{\delta}{\delta G_{sr}(y_1,x_1)}\Big(\delta^{At}_{Bu}\gamma^\mu_{BC}G_{CD}(y,x)\gamma^\nu_{DA}\delta^{(4)}(x-x_2)\delta^{(4)}(y-y_2)\\
&\qquad + G_{AB}(x,y)\gamma^\mu_{BC}\delta^{Ct}_{Du}\gamma^\nu_{DA}\delta^{(4)}(y-x_2)\delta^{(4)}(x-y_2)\Big)D_{\mu\nu}(x-y) && \text{(39b)}\\
&= \frac{g^2}{2\mathrm{i}}\int_{x^\mu}\int_{y^\mu}\Big(\gamma^\mu_{uC}\delta^{Cs}_{Dr}\gamma^\nu_{Dt}\delta^{(4)}(x-x_2)\delta^{(4)}(y-y_2)\delta^{(4)}(y-y_1)\delta^{(4)}(x-x_1)\\
&\qquad + \delta^{As}_{Br}\gamma^\mu_{Bt}\gamma^\nu_{uA}\delta^{(4)}(y-x_2)\delta^{(4)}(x-y_2)\delta^{(4)}(x-y_1)\delta^{(4)}(y-x_1)\Big)D_{\mu\nu}(x-y) && \text{(39c)}\\
&= \frac{g^2}{2\mathrm{i}}\Big(\gamma^\mu_{us}\gamma^\nu_{rt}D_{\mu\nu}(x_1-y_1)+\gamma^\mu_{rt}\gamma^\nu_{us}D_{\mu\nu}(y_1-x_1)\Big)\delta^{(4)}(x_1-x_2)\delta^{(4)}(y_1-y_2) && \text{(39d)}\\
&= -\mathrm{i}g^2\gamma^\mu_{us}\gamma^\nu_{rt}D_{\mu\nu}(x_1-y_1)\delta^{(4)}(x_1-x_2)\delta^{(4)}(y_1-y_2) && \text{(39e)}
\end{aligned}
$$

where we used $D_{\mu\nu}(x) = D_{\nu\mu}(-x)$. Unlike the NJL model, we do not have a local interaction here, instead the interaction is mediated by a propagating photon. Therefore we must look at the index structure of eq. (39e) to find the momentum distribution:



(40)

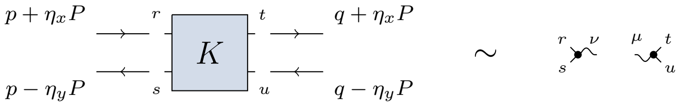
From momentum conservation we can see that the photon, when travelling from μ to ν, must have momentum $q-p$,

$$K^{(1.)}_{sr,tu}(p,q;P) = -\mathrm{i}g^2\gamma^\mu_{us}\gamma^\nu_{rt}D_{\mu\nu}(q-p) = -\mathrm{i}g^2\gamma^\mu_{rt}\gamma^\nu_{us}D_{\mu\nu}(p-q) \qquad (41)$$

Now onto the second term.

$$
\begin{aligned}
K^{(1.)}_{sr,tu} &= \frac{\mathrm{i}g^2}{2}\int_{x^\mu}\int_{y^\mu}\frac{\delta^2}{\delta G_{sr}(y_1,x_1)\delta G_{tu}(x_2,y_2)}\Big(G_{AB}(x,x)\gamma^\mu_{BA}G_{CD}(y,y)\gamma^\nu_{DC}\Big)D_{\mu\nu}(x-y) && \text{(42a)}\\
&= \frac{\mathrm{i}g^2}{2}\int_{x^\mu}\int_{y^\mu}\frac{\delta}{\delta G_{sr}(y_1,x_1)}\Big(\delta^{At}_{Bu}\gamma^\mu_{BA}G_{CD}(y,y)\gamma^\nu_{DC}\delta^{(4)}(x-x_2)\delta^{(4)}(x-y_2)\\
&\qquad + G_{AB}(x,x)\gamma^\mu_{BA}\delta^{Ct}_{Du}\gamma^\nu_{DC}\delta^{(4)}(y-x_2)\delta^{(4)}(y-y_2)\Big)D_{\mu\nu}(x-y) && \text{(42b)}\\
&= \frac{\mathrm{i}g^2}{2}\int_{x^\mu}\int_{y^\mu}\Big(\gamma^\mu_{ut}\delta^{Cs}_{Dr}\gamma^\nu_{DC}\delta^{(4)}(x-x_2)\delta^{(4)}(x-y_2)\delta^{(4)}(y-y_1)\delta^{(4)}(y-x_1)\\
&\qquad + \delta^{As}_{Br}\gamma^\mu_{BA}\gamma^\nu_{ut}\delta^{(4)}(y-x_2)\delta^{(4)}(y-y_2)\delta^{(4)}(x-y_1)\delta^{(4)}(x-x_1)\Big)D_{\mu\nu}(x-y) && \text{(42c)}\\
&= \frac{\mathrm{i}g^2}{2}\Big(\gamma^\mu_{ut}\gamma^\nu_{rs}D_{\mu\nu}(x_2-x_1)+\gamma^\mu_{rs}\gamma^\nu_{ut}D_{\mu\nu}(x_1-x_2)\Big)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_2-y_2) && \text{(42d)}\\
&= \mathrm{i}g^2\gamma^\mu_{ut}\gamma^\nu_{rs}D_{\mu\nu}(x_2-x_1)\delta^{(4)}(x_1-y_1)\delta^{(4)}(x_2-y_2) && \text{(42e)}
\end{aligned}
$$

Again, to find out how the momentum is distributed inside the photon propagator, we look at the index structure in eq. (42e):



(43)

From momentum conservation we can see that the photon, when travelling from μ to ν, must have momentum $-P$:

$$K^{(2.)}_{sr,tu}(p, q; P) = \mathrm{i}g^2 \gamma^\mu_{ut} \gamma^\nu_{rs} D_{\mu\nu}(-P) = \mathrm{i}g^2 \gamma^\mu_{rs} \gamma^\nu_{ut} D_{\mu\nu}(P) \tag{44}$$

So the complete QED BS kernel is given by

$$K_{sr,tu}(p, q; P) = -\mathrm{i}g^2 \gamma^\mu_{rt} \gamma^\nu_{us} D_{\mu\nu}(p - q) + \mathrm{i}g^2 \gamma^\mu_{rs} \gamma^\nu_{ut} D_{\mu\nu}(P) \tag{8.113}$$

We will meet the BS equation for QED again in section 10.

4 The Stability Criterion (8.7)

In a previous chapter we calculated the effective potential $V(\varphi_c)$. This potential has a physical interpretation: It yields the minimum value of the energy density among all normalized states $|\psi\rangle$ with a given value of the average $\varphi_c = \langle\psi|\varphi|\psi\rangle$. For a theory with a bilocal source $I(x, y)$, we also encountered the defining equations for the vacuum,

$$\frac{\delta\Gamma}{\delta\varphi_c} = 0, \qquad \frac{\delta\Gamma}{\delta G} = 0, \tag{8.1-2}$$

however, it can be shown that these conditions do not lead to a minimum of the potential, but rather to a saddle point. In this chapter, we will find a stability criterion with which we can guarantee that we find a minimum of the potential.

4.1 Classical Mechanics

Suppose we have a classical theory, described by an action functional,

$$S = \int \mathrm{d}t L(q, \dot{q}, t), \qquad L = \sum_{i=1}^{N} \frac{m}{2} \dot{q}_i^2 - U(q_i) \tag{8.119-120}$$

We are describing N particles of mass m that interact via a potential U. We get the equations of motion via Hamilton's principle,

$$\frac{\delta S}{\delta q_i} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial q_i} = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i}, \tag{8.121}$$

which leads to the N Euler–Lagrange equations. Let us consider a time-independent solution to the Euler–Lagrange equations, $q_i^{(0)}$. We want to investigate how variations from this static solutions behave: $q_i(t) = q_i^{(0)} + \Delta q_i(t)$. To do this, we do a Taylor expansion of eq. (8.121),

$$\begin{aligned} f(x+\epsilon) &= f(x) && + f'(x) && \epsilon && + \mathcal{O}(\epsilon^2) && (45) \\ \left.\frac{\delta S}{\delta q_i}\right|_{q_i(t)=q_i^{(0)}+\Delta q_i(t)} &= \underbrace{\left.\frac{\delta S}{\delta q_i}\right|_{q_i(t)=q_i^{(0)}}}_{0} && + \left.\frac{\delta^2 S}{\delta q_i(t) q_j(t')}\right|_{q_i(t)=q_i^{(0)}} && \Delta q_i(t') && + \mathcal{O}(\Delta q_i(t)^2) = 0 && (8.122) \end{aligned}$$

The first term vanishes, because $q^{(0)}$ satisfies the Euler–Lagrange equations. The second derivative of the action is a function of $\tau = t - t'$, therefore we can introduce its Fourier transform with only one frequency variable,

$$S_{ij}^{(2)}(\omega) = \int \mathrm{d}\tau \mathrm{e}^{\mathrm{i}\omega\tau} S_{ij}^{(2)}(\tau) \qquad (8.124)$$

and we also introduce the Fourier transform of q:

$$\Delta q_i(\omega) = \int \mathrm{d}t \mathrm{e}^{\mathrm{i}\omega t} \Delta q_i(t) \qquad (8.123)$$

Now eq. (8.122) can be written as a matrix equation,

$$S_{ij}^{(2)}(\omega)\Delta q_j(\omega) = 0 \qquad (8.125)$$

For the Lagrange function we specified earlier, $S^{(2)}$ can be explicitly stated:

$$S_{ij}^{(2)}(\omega) = \left[-m\omega^2\delta_{ij} + \frac{\delta^2 U}{\delta q_i \delta q_j}\right] \qquad (8.126)$$

Now we want to solve the equation eq. (8.125). To do this, we diagonalize the matrix U and write it as $U_{ij} = v^{(i)}\delta_{ij}$ with some eigenvalues $v^{(i)}$. The equation is fulfilled, if ω is equal to one of these eigenvalues, divided by the mass m. Therefore we can write a general solution as

$$\Delta q_i(\omega) = \sum_k C_{ik}\, \delta(\omega^2 - \tfrac{v^{(k)}}{m}) \qquad (46)$$

and the solution in real space contains exponential factors like

$$\Delta q_i(t) \sim \mathrm{e}^{\mathrm{i}\sqrt{\frac{v^{(k)}}{m}}t} \qquad (47)$$

From here we can see the criterion for stability: $\frac{v^{(k)}}{m}$ has to be real. Since U is hermitean, its eigenvalues $v^{(i)}$ are real. And this means that the stability criterion states that we must not have any tachyonic states (i.e. complex mass) in our system.

A Mathematical Details

A.1 Connection between the Classical Field and the Quantum Field

The classical field $\varphi_c(x)$ is defined as the solution of

$$\left.\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\right|_{\varphi=\varphi_c} = -J(x) \qquad (\to 3)$$

and the effective action $\Gamma[\varphi]$ is related to the generating functional of connected Green's functions via

$$\Gamma[\varphi] = W[J] - \varphi(x)J(x) \tag{48}$$

To make a connection between $\varphi(x)$ and $\varphi_c(x)$, we substitute eq. (48) into eq. (3),

$$\left.\frac{\delta W[J]}{\delta\varphi(x)}\right|_{\varphi=\varphi_c} - \left.\frac{\delta[\varphi(y)J(y)]}{\varphi(x)}\right|_{\varphi=\varphi_c} = -J(x) \tag{49}$$

We use the chain rule in the first term and the product rule in the second term,

$$\left.\frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\varphi(x)}\right|_{\varphi=\varphi_c} - \left.\frac{\delta J(y)}{\delta\varphi(x)}\varphi(y)\right|_{\varphi=\varphi_c} - J(y)\delta_{xy} = -J(x) \tag{50}$$

In the first two terms we can pull out a common factor and the third term cancels the term on the right-hand side of the equation:

$$\left\{\left[\frac{\delta W[J]}{\delta J(y)} - \varphi(y)\right]\frac{\delta J(y)}{\delta\varphi(x)}\right\}_{\varphi=\varphi_c} = 0 \tag{51}$$

For this equation to be valid in a general case, the terms in the square brackets must vanish. This entails,

$$\varphi_c(x) = \frac{\delta W[J]}{\delta J(x)} \tag{52}$$

The right-hand side can be evaluated using the generating functional:

$$\frac{\delta W[J]}{\delta J(x)} = \frac{\delta}{\mathrm{i}\delta J(x)}\ln\mathcal{Z}[J] = \frac{1}{\mathcal{Z}[J]}\frac{\delta\mathcal{Z}[J]}{\mathrm{i}\delta J(x)} = \langle 0|\,\mathsf{T}\varphi(x)\,|0\rangle_J \tag{53}$$

where the subscript J reminds us that we do not set the source equal to zero (which is otherwise done in order to get the Green's functions of the theory). Finally, we arrive at the result,

$$\varphi_c(x) = \langle 0|\,\mathsf{T}\varphi(x)\,|0\rangle \tag{54}$$

A.2 Free Propagator in Momentum Space

We start with

$$G^{(2)}_{0;rs,tu}(x_1y_1, x_2y_2) = -G_{rt}(x_1, x_2)G_{us}(y_2, y_1) \qquad (\rightarrow 8.75)$$

and want to find the corresponding expression in momentum space:

$$G^{(2)}_{0;rs,tu}(p, q; P) = \int_{z_1^\mu}\int_{z_2^\mu}\int_{z_3^\mu} \mathrm{e}^{\mathrm{i}(z_1p+z_2q+z_3P)}G^{(2)}_{0;rs,tu}(x_1y_1, x_2y_2) \tag{55}$$

$$= -\int_{z_1^\mu}\int_{z_2^\mu}\int_{z_3^\mu} \mathrm{e}^{\mathrm{i}(z_1p+z_2q+z_3P)}G_{rt}(x_1, x_2)G_{us}(y_2, y_1) \tag{56}$$

We transform both propagators G into momentum space,

$$G_{rt}(x_1, x_2) = \int_{\tilde{p}^\mu} \mathrm{e}^{-\mathrm{i}(x_1-x_2)\tilde{p}}G_{rt}(\tilde{p}) \tag{57}$$

such that:

$$G^{(2)}_{0;rs,tu}(p,q;P) = -\int_{z_1^\mu}\int_{z_2^\mu}\int_{z_3^\mu}\int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} \mathrm{e}^{\mathrm{i}(z_1 p + z_2 q + z_3 P) - \mathrm{i}(x_1 - x_2)\tilde{p} - \mathrm{i}(y_2 - y_1)\tilde{q}} G_{rt}(\tilde{p}) G_{us}(\tilde{q}) \quad (58)$$

We replace the z-terms in the exponential,

$$x_1 p - y_1 p + y_2 q - x_2 q + \eta_x (x_1 - x_2) P + \eta_y (y_1 - y_2) P - (x_1 - x_2)\tilde{p} - (y_2 - y_1)\tilde{q} \quad (59)$$

Let us rearrange the terms:

$$\underbrace{(x_1 - x_2)}_{w_1}(p + \eta_x P - \tilde{p}) + \underbrace{(y_2 - y_1)}_{w_2}(q - \eta_y P - \tilde{q}) + \underbrace{(x_2 - y_1)}_{w_3}(p - q) \quad (60)$$

which means we can integrate over $w_1 = x_1 - x_2$, $w_2 = y_2 - y_1$ and $w_3 = x_2 - y_1$ instead of over z_1, z_2, z_3:

$$G^{(2)}_{0;rs,tu}(p,q;P) = -\int_{w_1^\mu}\int_{w_2^\mu}\int_{w_3^\mu}\int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} \mathrm{e}^{w_1(p + \eta_x P - \tilde{p}) + w_2(q - \eta_y P - \tilde{q}) + w_3(p - q)} G_{rt}(\tilde{p}) G_{us}(\tilde{q}) \quad (61)$$

The integrations over $w_{1,2,3}$ yield delta functions,

$$G^{(2)}_{0;rs,tu}(p,q;P) = -(2\pi)^{12} \int_{\tilde{p}^\mu}\int_{\tilde{q}^\mu} \delta^{(4)}(p + \eta_x P - \tilde{p})\,\delta^{(4)}(q - \eta_y P - \tilde{q})$$
$$\delta^{(4)}(p - q) G_{rt}(\tilde{p}) G_{us}(\tilde{q}) \quad (62)$$

Due to the delta functions, we can easily perform the integrations over $\tilde{p}$ and $\tilde{q}$ and we finally arrive at the desired expression:

$$G^{(2)}_{0;rs,tu}(p,q;P) = -(2\pi)^4 \delta^{(4)}(p - q) G_{rt}(p + \eta_x P) G_{us}(p - \eta_y P) \quad (\to 8.80)$$

A.3 BS Wave Function with Translational Invariance

The translation operator acts in the following way on a scalar field,

$$\varphi(x + a) = \mathrm{e}^{\mathrm{i}\hat{P}a} \varphi(x) \mathrm{e}^{-\mathrm{i}\hat{P}a} = U(a)\varphi(x)U^\dagger(a) \quad (63)$$

The bound state $|P_b, i\rangle$ is an eigenstate of the translation operator[7],

$$U(a)\,|P_b, i\rangle = \mathrm{e}^{\mathrm{i}\hat{P}a}\,|P_b, i\rangle = \mathrm{e}^{\mathrm{i}P_b a}\,|P_b, i\rangle \quad (64)$$

We start the calculation by inserting an identity relation $\mathbb{1} = U^\dagger U$ into the BS wave function,

$$\chi_{i,rs}(x, y, P_b) = \langle 0|\mathsf{T} U^\dagger(a) U(a) \varphi_r(x) U^\dagger(a) U(a) \varphi_s(y) U^\dagger(a) U(a) |P_b, i\rangle \quad (65)$$

Now we use eqs. (63) and (64) and the fact that the vacuum state is a state with zero momentum to get:

$$\chi_{i,rs}(x, y, P_b) = \mathrm{e}^{-\mathrm{i}0a}\,\langle 0|\mathsf{T}\varphi_r(x + a)\varphi_s(y + a)|P_b, i\rangle\,\mathrm{e}^{\mathrm{i}P_b a} \quad (66)$$

[7]Because this bound state vector has a well-defined momentum.

We will now use a specific $a := -X$, where $X = \eta_x x + \eta_y y$ is the center-of-mass coordinate. With this choice for a, the arguments of the fields are:

$$x + a = x - \eta_x x - \eta_y y = (\eta_x + \eta_y)x - \eta_x x - \eta_y y = \eta_y(x - y), \tag{67}$$
$$y + a = y - \eta_x x - \eta_y y = (\eta_x + \eta_y)y - \eta_x x - \eta_y y = \eta_x(y - x). \tag{68}$$

So we write the BS wave function as

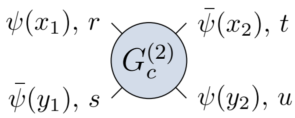
$$\chi_{i,rs}(x, y, P_b) = \langle 0|\mathsf{T}\varphi_r(\eta_y z)\varphi_s(-\eta_x z)|P_b, i\rangle\, \mathrm{e}^{-\mathrm{i}P_b X} =: \chi_{i,rs}(z, P_b)\mathrm{e}^{-\mathrm{i}P_b X} \qquad (\rightarrow 8.84)$$

and the BS wave function in momentum space is

$$\chi_{i,rs}(x, y, P_b) = \mathrm{e}^{-\mathrm{i}P_b X} \int_{p^\mu} \mathrm{e}^{-\mathrm{i}p(x-y)}\, \chi_{i,rs}(p, P_b). \qquad (\rightarrow 8.88)$$

A.4 Bound States are Poles in Green's Functions

In this section we show that a mesonic pole appears in a Green's function of two fermions and two antifermions. A standard reference is Ref. [8]. Consider a fermionic 4-point Green function defined as



$$G^{(2)}_{c;rs,ut}(x_1y_1, y_2x_2) = \langle 0|\mathsf{T}\psi_r(x_1)\bar\psi_s(y_1)\bar\psi_t(x_2)\psi_u(y_2)|0\rangle \qquad (8.98)$$

where $\psi(x_i)$ and $\bar\psi(x_i) = \psi^\dagger(x_i)\gamma^0$ are fermionic field operators, corresponding to the annihilation and creation of a fermionic particle (e.g. a quark), respectively. $x_i = (t, \mathbf{x}_i)$ is the four-position vector, T represents time-ordering and $|0\rangle$ is the vacuum ground state. We assume that the fermions' positions can be split up into a center-of-mass movement and a relative movement. The available states in our Hilbert space are the vacuum $|0\rangle$, the one-particle states $|k\rangle$ with momentum $\mathbf{k}$ (and energy $\omega_\mathbf{k} = +\sqrt{\mathbf{k}^2 + m^2}$) and multi-particle states $|k, n\rangle$ with center-of-mass momentum $\mathbf{k}$ and other (discrete and/or continuous) quantum numbers n. We insert a completeness relation [9] to perform a spectral decomposition,

$$1 = |0\rangle\langle 0| + \int \widetilde{\mathrm{d}k}\, |k\rangle\langle k| + \sum\!\!\!\!\!\!\int_n \widetilde{\mathrm{d}k}\, |k, n\rangle\langle k, n|\,, \qquad \widetilde{\mathrm{d}k} \equiv \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{k}} \tag{69}$$

into the 4-point Green function, Equation (8.98). In the scope of this work, we are interested in mesons, which consist of a quark anti-quark pair. Therefore, we will only consider the multi-particle terms in Equation (69) that can be brought to the vacuum's quantum numbers via one ψ and one $\bar\psi$. Also, we ignore the $|0\rangle\langle 0|$ contribution, because we want to describe a bound state and the $|0\rangle\langle 0|$ contribution would lead only to quark propagation between $x_{1(2)}$ and $y_{1(2)}$. The time-ordered product simplifies to,

$$\begin{aligned} G^{(2)}_{\text{meson}}(x_1y_1, y_2x_2) = \sum\!\!\!\!\!\!\int_n \widetilde{\mathrm{d}k}\, &\langle 0|\,\mathsf{T}\psi_{x_1}\bar\psi_{y_1}\,|k, n\rangle\langle k, n|\,\mathsf{T}\bar\psi_{x_2}\psi_{y_2}\,|0\rangle \\ &\times \Theta(\min(x_1^0, y_1^0) - \max(x_2^0, y_2^0)), \end{aligned} \tag{70}$$

where $\Theta(x)$ is the Heaviside unit step function, which will be adressed in Equations (75) to (78) and appears to ensure time ordering. Its argument is chosen such that the constituent particles of the initial state are chronologically earlier than those of the final state. We introduce the Bethe-Salpeter wave function χ and its conjugate via,

$$\begin{aligned}\chi(x_1, y_1; \mathbf{k}, n) &= \langle 0|\, \mathsf{T}\psi(x_1)\bar{\psi}(y_1)\, |k, n\rangle\,, \\ \bar{\chi}(x_2, y_2; \mathbf{k}, n) &= \langle k, n|\, \mathsf{T}\bar{\psi}(x_2)\psi(y_2)\, |0\rangle\,,\end{aligned} \tag{71}$$

and will consider the meson state from this point on as a one-particle state with three-momentum $\mathbf{P}_b$ (i.e. $|k,n\rangle \to |P_b\rangle$ and $(\mathbf{k}, n) \to (\mathbf{P}_b)$),

$$\chi(x_1, y_1; \mathbf{P}_b) = \langle 0|\, \mathsf{T}\psi(x_1)\bar{\psi}(y_1)\, |P_b\rangle\,, \tag{72}$$

We can reduce Equation (72) to only depend on one degree of freedom, the relative coordinates $z_{1,2}$, using how the translation operator acts on a scalar field, $\varphi(x+a) = \exp(\mathrm{i}\hat{P}a)\varphi(x)\exp(-\mathrm{i}\hat{P}a)$,

$$\begin{aligned}\chi(x_1, y_1; \mathbf{P}_b) &= \chi(z_1; \mathbf{P}_b)\mathrm{e}^{-\mathrm{i}P_b\cdot Z_1}, \qquad & z_1 &= x_1 - y_1;\ Z_1 = \eta_x x_1 + \eta_y y_1 \\ \bar{\chi}(x_2, y_2; \mathbf{P}_b) &= \chi(z_2; \mathbf{P}_b)\mathrm{e}^{\mathrm{i}P_b\cdot Z_2}, & z_2 &= y_2 - x_2;\ Z_2 = \eta_x x_2 + \eta_y y_2\end{aligned}$$

where $\hat{P}^\mu$ is the four-momentum operator acting as the generator for spacetime translations and we used that the meson state is an eigenvector of this momentum operator with eigenvalues P_M^μ. We use the following conventions of a Fourier transformation,

$$f(p) = \int\limits_{x^\mu} f(x)\, \mathrm{e}^{\mathrm{i}p\cdot x} = \int \mathrm{d}^4x f(x)\, \mathrm{e}^{\mathrm{i}p\cdot x}, \quad f(x) = \int\limits_{p^\mu} f(p)\, \mathrm{e}^{-\mathrm{i}p\cdot x} = \int \tfrac{\mathrm{d}^4p}{(2\pi)^4} f(p)\, \mathrm{e}^{-\mathrm{i}p\cdot x}, \tag{73}$$

where we denote momentum space integrals as $\int_{p^\mu} \equiv \int \mathrm{d}^4p(2\pi)^{-4}$ and coordinate space integrals as $\int_{x^\mu} \equiv \int \mathrm{d}^4x$ for brevity. The amplitudes are transformed to momentum space via

$$\chi(z_1; \mathbf{P}_M) = \int\limits_{p^\mu} \mathrm{e}^{-\mathrm{i}pz_1}\, \chi(p, \mathbf{P}_M), \qquad \bar{\chi}(z_2; \mathbf{P}_M) = \int\limits_{q^\mu} \mathrm{e}^{-\mathrm{i}qz_2}\, \bar{\chi}(q, \mathbf{P}_M). \tag{74}$$

Now we turn to the Heaviside step function in Equation (70), ensuring time-ordering in the original Green function. With the help of the identities,

$$\max(a,b) = \frac{a+b}{2} + \frac{|a-b|}{2}, \qquad \min(a,b) = \frac{a+b}{2} - \frac{|a-b|}{2}, \tag{75}$$

the argument of the Heaviside step function can be expressed as,[8]

$$\min(x_1^0, y_1^0) - \max(y_2^0, x_2^0) = Z_1^0 - Z_2^0 - \frac{1}{2}|z_1^0| - \frac{1}{2}|z_2^0| + \frac{1}{2}(\eta_y - \eta_x)(z_1^0 + z_2^0). \tag{76}$$

We use the following integral representation of the Heaviside step function,

$$\Theta(x) = \frac{\mathrm{i}}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}w \frac{\mathrm{e}^{-\mathrm{i}xw}}{w + \mathrm{i}\epsilon}, \quad \epsilon \to 0^+, \tag{77}$$

[8] We use that $x_1 = \eta_y z_1 + Z_1$, $y_1 = -\eta_x z_1 + Z_1$, $x_2 = -\eta_y z_2 + Z_2$, $y_2 = \eta_x z_2 + Z_2$.

so that we can write the step function as,

$$\Theta\Big(\min(x_1^0,x_2^0)-\max(y_1^0,y_2^0)\Big)=\frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}w\frac{\mathrm{e}^{-\mathrm{i}w(Z_1^0-Z_2^0+\xi^0)}}{w+\mathrm{i}\epsilon}, \tag{78}$$

where we set $\xi^0 := -\frac{1}{2}|z_1^0| - \frac{1}{2}|z_2^0| + \frac{1}{2}(\eta_y-\eta_x)(z_1^0+z_2^0)$ for brevity. Let us insert Equation (78) into the meson 4-point Green function (70),

$$\begin{aligned} G^{(2)}_{\text{meson}}(x_1y_1,y_2x_2) = \frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}w\int\widetilde{\mathrm{d}P_b}\int\limits_{p^\mu}\int\limits_{q^\mu}\frac{\chi(p,\mathbf{P}_b)\bar\chi(q,\mathbf{P}_b)}{w+\mathrm{i}\epsilon} \\ \times\,\mathrm{e}^{-\mathrm{i}P_M(Z_1-Z_2)}\mathrm{e}^{-\mathrm{i}p\cdot z_1}\mathrm{e}^{-\mathrm{i}q\cdot z_2}\mathrm{e}^{-\mathrm{i}w(Z_1^0-Z_2^0+\xi^0)}. \end{aligned} \tag{79}$$

For the last step, we transform this Green function to momentum space ($z_3 = Z_1 - Z_2$),

$$G^{(2)}_{\text{meson}}(k,k';K) = \int\limits_{z_1^\mu}\int\limits_{z_2^\mu}\int\limits_{z_3^\mu}\mathrm{e}^{\mathrm{i}k\cdot z_1}\mathrm{e}^{\mathrm{i}k'\cdot z_2}\mathrm{e}^{\mathrm{i}K\cdot z_3}G^{(2)}_{\text{meson}}(x_1y_1,y_2x_2), \tag{80}$$

where we chose the three translational invariant coordinates to be z_1, z_2 and z_3. If we only look at the $\mathrm{d}z_3^0$ integration, we pick up the following arguments,

$$\int\limits_{z_3^0}\mathrm{e}^{-\mathrm{i}z_3^0(P_b^0+w-K^0)} = 2\pi\,\delta\Big(w-(K^0-P_b^0)\Big),\quad P_b^0=\omega_{\mathbf{P}_b}. \tag{81}$$

This shows that after the w integration, a pole emerges, when the total energy K^0 transferred through a Green function equals the energy P_M^0 of a bound state in the same Hilbert space. From here on we drop the label "meson". We are left with,

$$\begin{aligned} G^{(2)}(k,k';K) = \mathrm{i}\int\limits_{z_1^\mu}\int\limits_{z_2^\mu}\int\limits_{\mathbf{z}_3}\int\widetilde{\mathrm{d}P_b}\int\limits_{p^\mu}\int\limits_{q^\mu}\frac{\chi(p,\mathbf{P}_b)\bar\chi(q,\mathbf{P}_b)}{K^0-\omega_{\mathbf{P}_b}+\mathrm{i}\epsilon}\mathrm{e}^{\mathrm{i}(k\cdot z_1+k'\cdot z_2-p\cdot z_1-q\cdot z_2)} \\ \times\,\mathrm{e}^{-\mathrm{i}\mathbf{P}_b\mathbf{z}_3}\mathrm{e}^{-\mathrm{i}\xi^0(K^0-\omega_{\mathbf{P}_b})}\mathrm{e}^{\mathrm{i}\mathbf{K}\mathbf{z}_3}. \end{aligned}$$

Next, we perform the $\mathrm{d}\mathbf{z}_3$ integration, which leads to a delta function $\delta^{(3)}(\mathbf{K}-\mathbf{P}_b)$. The factors of (2π) cancel the ones of the tilded integration measure, and integrating over $\mathbf{P}_b$ yields,

$$G^{(2)}(k,k';K) = \mathrm{i}\int\limits_{z_1^\mu}\int\limits_{z_2^\mu}\int\limits_{p^\mu}\int\limits_{q^\mu}\frac{1}{2\omega_{\mathbf{P}_b}}\frac{\chi(p,\mathbf{K})\bar\chi(q,\mathbf{K})}{K^0-\omega_{\mathbf{K}}+\mathrm{i}\epsilon}\mathrm{e}^{\mathrm{i}z_1\cdot(k-p)}\mathrm{e}^{\mathrm{i}z_2\cdot(q-k')}\mathrm{e}^{-\mathrm{i}\xi^0(K^0-\omega_{\mathbf{K}})}.$$

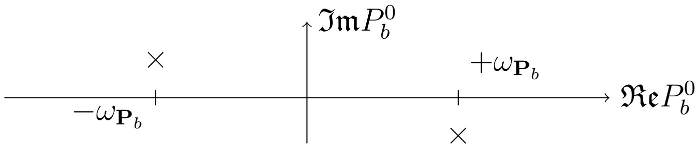
At this point, the d^4z_1 and d^4z_2 integrations cannot be performed to obtain Dirac delta functions, due to the presence of ξ^0, which contains x_1^0, y_1^0, x_2^0 and y_2^0. Only the spatial $\mathrm{d}^3\mathbf{z}_1$ and $\mathrm{d}^3\mathbf{z}_2$ integrations are traivial. But, if we take the limit $K^0\to\omega_{\mathbf{K}}$, that is only consider momenta with energies near a bound-state energy, the exponential factor containing ξ^0 drops out. Equivalently, one could expand the exponential $\exp(-\mathrm{i}\xi^0(K^0-\omega_{\mathbf{K}}))$ into a Taylor series. Taking only the first term would lead to the same results as below and every additional term contributes to scattering terms, which we do not consider here because they would be regular (i.e. of order $\mathcal{O}((K^0-\omega_{\mathbf{K}})^n)$).

$$\begin{aligned}\lim_{K^0\to\omega_{\mathbf{K}}} G^{(2)}(k,k';K) &= \mathrm{i}\int_{z_1^\mu}\int_{z_2^\mu}\int_{p^\mu}\int_{q^\mu} \frac{1}{2\omega_{\mathbf{K}}}\frac{\chi(p,\mathbf{K})\bar{\chi}(q,\mathbf{K})}{K^0-\omega_{\mathbf{K}}+\mathrm{i}\epsilon}\mathrm{e}^{\mathrm{i}z_1\cdot(k-p)}\mathrm{e}^{\mathrm{i}z_2\cdot(q-k')} \\ &= \mathrm{i}\int_{p^\mu}\int_{q^\mu}\frac{1}{2\omega_{\mathbf{K}}}\frac{\chi(p,\mathbf{K})\bar{\chi}(q,\mathbf{K})}{K^0-\omega_{\mathbf{K}}+\mathrm{i}\epsilon}(2\pi)^8\delta^{(4)}(p-k)\delta^{(4)}(q-k') \\ &= \frac{\mathrm{i}}{2\omega_{\mathbf{K}}}\frac{\chi(k,\mathbf{K})\bar{\chi}(k',\mathbf{K})}{K^0-\omega_{\mathbf{K}}+\mathrm{i}\epsilon}. \end{aligned} \tag{82}$$

The same reasoning leads to similar terms for the case of the bound-state's antiparticle with negative energy. Note that the $\omega_{\mathbf{k}}$ in eq. (69) is negative. We write it as $-\omega_{\mathbf{k}}$ with $\omega_{\mathbf{k}}$ positive:

$$\begin{aligned}\lim_{K^0\to-\omega_{\mathbf{K}}} G^{(2)}(k,k';K) &= \mathrm{i}\int_{x^\mu}\int_{y^\mu}\int_{p^\mu}\int_{q^\mu} \frac{1}{2(-\omega_{\mathbf{K}})}\frac{\chi(p,\mathbf{K})\bar{\chi}(q,\mathbf{K})}{K^0+\omega_{\mathbf{K}}-\mathrm{i}\epsilon}\mathrm{e}^{\mathrm{i}x\cdot(k-p)}\mathrm{e}^{\mathrm{i}y\cdot(q-k')} \\ &= -\mathrm{i}\int_{p^\mu}\int_{q^\mu}\frac{1}{2\omega_{\mathbf{K}}}\frac{\chi(p,\mathbf{K})\bar{\chi}(q,\mathbf{K})}{K^0+\omega_{\mathbf{K}}-\mathrm{i}\epsilon}(2\pi)^8\delta^{(4)}(p-k)\delta^{(4)}(q-k') \\ &= -\frac{\mathrm{i}}{2\omega_{\mathbf{K}}}\frac{\chi(k,\mathbf{K})\bar{\chi}(k',\mathbf{K})}{K^0+\omega_{\mathbf{K}}-\mathrm{i}\epsilon}. \end{aligned} \tag{83}$$

Note that we have "$-\mathrm{i}\epsilon$" here. If we consider the poles of a time-ordered Green's function (Feynman prescription), they are positioned like this:



For the particle-case, we want to integrate above the pole at $P_b^0=\omega_{\mathbf{P}_b}$. Therefore we shift the pole position downwards, i.e. $\omega_{\mathbf{P}_b}\to\omega_{\mathbf{P}_b}-\mathrm{i}\epsilon$. This means that the pole term changes in the following way:

$$\frac{1}{P_b^0-\omega_{\mathbf{P}_b}}\to\frac{1}{P_b^0-(\omega_{\mathbf{P}_b}-\mathrm{i}\epsilon)}=\frac{1}{P_b^0-\omega_{\mathbf{P}_b}+\mathrm{i}\epsilon}. \tag{84}$$

Similarly, for the aniparticle-case, we get the "$-\mathrm{i}\epsilon$" instead. Adding both particle and antiparticle contributions leads to (ignoring terms $\mathcal{O}(\epsilon^2)$)

$$\begin{aligned} G^{(2)}(k,k';K)\Big|_{\text{poles}} &= \frac{\mathrm{i}\chi(k,\mathbf{K})\bar{\chi}(k',\mathbf{K})}{2\omega_{\mathbf{K}}}\left[\frac{1}{K^0-\omega_{\mathbf{K}}+\mathrm{i}\epsilon}-\frac{1}{K^0+\omega_{\mathbf{K}}-\mathrm{i}\epsilon}\right] \\ &= \frac{\mathrm{i}\chi(k,\mathbf{K})\bar{\chi}(k',\mathbf{K})}{2\omega_{\mathbf{K}}}\left[\frac{K^0+\omega_{\mathbf{K}}-\mathrm{i}\epsilon-K^0+\omega_{\mathbf{K}}-\mathrm{i}\epsilon}{(K^0)^2+K^0\omega_{\mathbf{K}}-\mathrm{i}K^0\epsilon-K^0\omega_{\mathbf{K}}+\mathrm{i}\epsilon\omega_{\mathbf{K}}-\omega_{\mathbf{K}}^2+\mathrm{i}\epsilon K^0+\mathrm{i}\epsilon\omega_{\mathbf{K}}}\right] \\ &= \frac{\mathrm{i}\chi(k,\mathbf{K})\bar{\chi}(k',\mathbf{K})}{2\omega_{\mathbf{K}}}\left[\frac{2\omega_{\mathbf{K}}}{(K^0)^2-\omega_{\mathbf{K}}^2+2\mathrm{i}\epsilon\omega_{\mathbf{K}}}\right] \quad (\omega_{\mathbf{K}}\text{ is constant}) \\ &= \frac{\mathrm{i}\chi(k,\mathbf{K})\bar{\chi}(k',\mathbf{K})}{(K^0)^2-\omega_{\mathbf{K}}^2+\mathrm{i}\epsilon} \end{aligned} \tag{85}$$

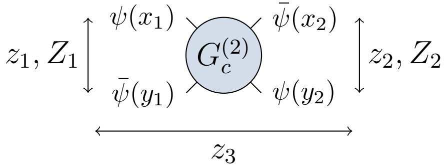
We can use the explicit expression for $\omega_{\mathbf{K}} = (\mathbf{K}^2 - m_{P_b}^2)^{1/2}$ and write $m_{P_b}^2 = P_b^2 = P_b^\mu P_{b\mu}$,

$$G^{(2)}(k, k'; K)\Big|_{\text{poles}} = \mathrm{i}\frac{\chi(k, \mathbf{K})\bar{\chi}(k', \mathbf{K})}{K^2 - P_b^2 + \mathrm{i}\epsilon} \tag{86}$$

Equation (86) shows, that for energies near a bound state energy, the 4-point Green function can be represented by a diverging term, containing the Bethe-Salpeter amplitude and its conjugate.

A.5 Relative Positions and Momenta

This section gives an overview of all the different x_i and z_j we use in this summary. We start with $\{x_1, y_1, y_2, x_2\}$, which are the individual positions of the constituent particles (note: fermionic case!),



(87)

Next, we introduced the relative position vectors $z_{1,2}$ and center-of-mass vectors $Z_{1,2}$ between the initial and final states:

$$\left\{\begin{array}{l} z_1 = x_1 - y_1 \\ Z_1 = \eta_x x_1 + \eta_y y_1 \\ z_2 = y_2 - x_2 \\ Z_2 = \eta_x x_2 + \eta_y y_2 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} x_1 = \eta_y z_1 + Z_1 \\ y_1 = -\eta_x z_1 + Z_1 \\ y_2 = \eta_x z_2 + Z_2 \\ x_2 = -\eta_y z_2 + Z_2 \end{array}\right\} \tag{88}$$

Due to translation invariance, we can reduce our four degrees of freedom to only three degrees of freedom. To do this, we introduce z_3:

$$z_3 = Z_1 - Z_2. \tag{89}$$

Finally, when we go to momentum space, the conjugated momenta are chosen such that:

$$z_1 \leftrightarrow p, \quad z_2 \leftrightarrow q, \quad z_3 \leftrightarrow P. \tag{90}$$

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