Chirality in the EM Field Strenght Tensor

Stephan Hübsch*

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Abstract

Chirality is a concept that appears in an (A, B) representation of the Lorentz group for $A \neq B$. The electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ transforms under the $(1, 0) \oplus (0, 1)$ representation. Therefore it should exhibit some kind of chirality. We first investigate the spin-$\frac{1}{2}$ case in Section 1 and transition to the spin-1 case in Section 2. In Section 3 we discuss the Riemann–Silberstein vectors, which will help distingusih different chiralities in the spin-1 case.

Contents

*huebsch.s.aa@m.titech.ac.jp

1 Dirac and Weyl Spinors

A Dirac spinor Ψ transforms under the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group. It has four components, so it should be possible to separate left-chiral from right-chiral components. Its equation of motion is the Dirac equation:

$$(\mathrm{i}\partial\!\!\!/ - m)\Psi = 0, \quad \partial\!\!\!/ = \gamma^\mu \partial_\mu. \tag{1}$$

If we consider massless ($m=0$) particles and use the chiral representation for the gamma matrices as well as the conventient notation of $\overline{\sigma}^\mu$,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \sigma^\mu = \begin{pmatrix} \mathbb{1} \\ \sigma^k \end{pmatrix}, \quad \overline{\sigma}^\mu = \begin{pmatrix} \mathbb{1} \\ -\sigma^k \end{pmatrix}, \tag{2}$$

we can split the Dirac equation into two independent equations for two-component spinors:

$$\mathrm{i}\partial\!\!\!/\Psi = \mathrm{i}\begin{pmatrix} 0 & \sigma^\mu\partial_\mu \\ \overline{\sigma}^\mu\partial_\mu & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \quad \rightarrow \quad \begin{cases} \mathrm{i}\sigma^\mu\partial_\mu\psi_R = 0 \\ \mathrm{i}\overline{\sigma}^\mu\partial_\mu\psi_L = 0 \end{cases}. \tag{3}$$

Using $p^k = -\mathrm{i}\partial^k$, we can write these two equations as:

$$\mathrm{i}\frac{\partial\psi_R}{\partial t} + \mathrm{i}\sigma^k\partial_k\psi_R = 0 \quad \rightarrow \quad \mathrm{i}\frac{\partial\psi_R}{\partial t} = (\boldsymbol{\sigma}\cdot\mathbf{p})\psi_R \tag{4a}$$

$$\mathrm{i}\frac{\partial\psi_L}{\partial t} - \mathrm{i}\sigma^k\partial_k\psi_L = 0 \quad \rightarrow \quad \mathrm{i}\frac{\partial\psi_L}{\partial t} = -(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_L \tag{4b}$$

ψ_L and ψ_R are called "Weyl spinors". As spin-$\frac{1}{2}$ objects, they have $(2s+1)=2$ components. If we use the rather unusual[1] notation of $\psi_R \rightarrow \psi_+$ and $\psi_L \rightarrow \psi_-$, we can write both equations in (4) into one equation:

$$\boxed{\mathrm{i}\frac{\partial\psi_\pm}{\partial t} = \pm(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_\pm} \tag{5}$$

We have successfully identified the chiral components of the Dirac spinor and their equations of motion. In the next sections, we try to find similar results for the $(1,0)\oplus(0,1)$ case.

2 Dual Tensor

The $(1,0)$ and $(0,1)$ representations can be differentiated with the hodge dual field strength tensor, $\widetilde{F} = \star F$. It can be calculated as $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. In a four-dimensional Euclidean spacetime, $\star\star = 1$, but for a Minkowski spacetime, $\star\star = -1$ [1]. We will stick to Minkowski spacetime, so one could impose one of two conditions:

$$F^{\mu\nu} = \mathrm{i}\widetilde{F}^{\mu\nu} \quad \text{or} \quad F^{\mu\nu} = -\mathrm{i}\widetilde{F}^{\mu\nu} \tag{6}$$

These conditions are preserved by Lorentz transformations and correspond to the $(1,0)$ or $(0,1)$ representations[2]. The electromagnetic field strength tensor and its hodge dual

[1] But useful for comparing to the spin-1 case!

[2] They are the fundamental representations of $SO(3,\mathbb{C})$.

are given in terms of electric and magnetic fields by,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad \widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{7}$$

This means that the duality conditions lead to different conditions for the electric and magnetic fields:

$$F = \mathrm{i}\widetilde{F} \;\rightarrow\; \mathbf{E} = \mathrm{i}\mathbf{B} \qquad \text{or} \qquad F = -\mathrm{i}\widetilde{F} \;\rightarrow\; \mathbf{E} = -\mathrm{i}\mathbf{B}. \tag{8}$$

So in order to follow these conditions and have orthogonal fields, one possible solution[3] would be $\mathbf{E} = \left(\begin{smallmatrix} 1 \\ \mathrm{i} \\ 0 \end{smallmatrix}\right) \mathrm{e}^{i(kz-\omega t)}$, which for $k > 0$ would be a left-circular wave propagating in positive z-direction. We then calculate the magnetic fields in both duality cases and check whether Maxwell's equations are fulfilled: the unit vectors for the electric and magnetic field as well as for the direction of propagation have to obey $\hat{\mathbf{e}}_E \times \hat{\mathbf{e}}_B = \hat{\mathbf{e}}_k \overset{\text{here}}{=} \hat{\mathbf{e}}_z$:

$$\mathfrak{Re}\mathbf{E} = \begin{pmatrix} \cos(kz-\omega t) \\ \sin(kz-\omega t) \\ 0 \end{pmatrix} \quad \begin{array}{ll} \overset{\mathbf{E}=\mathrm{i}\mathbf{B}}{\longrightarrow} \quad \mathfrak{Re}\mathbf{B} = \begin{pmatrix} \sin(kz-\omega t) \\ \cos(kz-\omega t) \\ 0 \end{pmatrix}, & \hat{\mathbf{e}}_E \times \hat{\mathbf{e}}_B = \hat{\mathbf{e}}_z \\ \overset{\mathbf{E}=-\mathrm{i}\mathbf{B}}{\longrightarrow} \quad \mathfrak{Re}\mathbf{B} = \begin{pmatrix} -\sin(kz-\omega t) \\ -\cos(kz-\omega t) \\ 0 \end{pmatrix}, & \hat{\mathbf{e}}_E \times \hat{\mathbf{e}}_B = -\hat{\mathbf{e}}_z \end{array} \tag{9}$$

We can see that the case $\mathbf{E} = -\mathrm{i}\mathbf{B}$ is only consistent for $k < 0$, which means, we are actually describing a right-circular wave! We conclude: The condition $F = \mathrm{i}\widetilde{F}$ leads to left-circular polarized waves, whereas the condition $\widetilde{F} = -\mathrm{i}F$ leads to right-circular polarized waves.

We have successfully identified how the $(1, 0)$ and $(0, 1)$ representations lead to a chiral phenomenon. In the next section, we show that those left- and right-circular waves also follow different equations of motion, similar to left- and right-chiral Weyl spinors.

3 Riemann–Silberstein Vectors

For electrodynamics in a vacuum, the equations of motion are $\partial_\mu F^{\mu\nu} = 0$ and $\partial_\mu \widetilde{F}^{\mu\nu} = 0$. In three-notation, these are the four Maxwell equations. For the following section, we need only two of them,

$$\epsilon_{ijk}\partial_j E_k = -\dot{B}_i, \quad \epsilon_{ijk}\partial_j B_k = \dot{E}_i. \tag{10}$$

If we add or subtract the first equation to i-times the second one, we get

$$\epsilon_{ijk}\partial_j(E_k \pm \mathrm{i}B_k) = -\dot{B}_i \pm \mathrm{i}\dot{E}_i = \pm\mathrm{i}(\dot{E}_i \pm \mathrm{i}\dot{B}_i). \tag{11}$$

We define the three-component *Riemann–Silberstein vectors*: $\psi_k^\pm = E_k \pm \mathrm{i}B_k$. Using $p_k = -\mathrm{i}\partial_k$, we can write these two equations as,

$$\epsilon_{ijk} p_j \psi_k^\pm = \pm\frac{\partial \psi_i^\pm}{\partial t}. \tag{12}$$

[3] Assume that the amplitude of the wave $|E_0| = 1$ for simplicity.

We now mutiply with a factor of i, exchange two indices in the epsilon tensor and make use of the spin-1 matrices $(\Sigma_i)_{jk} = -\mathrm{i}\epsilon_{ijk}$[4]:

$$-\mathrm{i}\epsilon_{jik}p_j\psi_k^\pm = \pm\mathrm{i}\frac{\partial\psi_i^\pm}{\partial t} \quad\rightarrow\quad (\Sigma_j)_{ik}p_j\psi_k^\pm = \pm\mathrm{i}\frac{\partial\psi_i^\pm}{\partial t}. \tag{13}$$

If we switch from index notation to vector notation, we get a similar equation like the one describing spinors in Equation (5):

$$\boxed{\mathrm{i}\frac{\partial\boldsymbol{\psi}^\pm}{\partial t} = \pm(\boldsymbol{\Sigma}\cdot\mathbf{p})\boldsymbol{\psi}^\pm} \qquad \text{compare to: } \mathrm{i}\frac{\partial\psi_\pm}{\partial t} = \pm(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_\pm. \tag{14}$$

In order to recognize the connection to chirality, we have to remember the conditions from the previous section:

$$\text{left-chiral: } \mathbf{E} = \mathrm{i}\mathbf{B}, \qquad \text{right-chiral: } \mathbf{E} = -\mathrm{i}\mathbf{B}. \tag{15}$$

If we only want to consider right-chiral waves, then we see that ψ^+ vanishes! So the remaining component of $F^{\mu\nu}$, namely ψ^-, is right-chiral. On the other hand, if we only want to consider left-chiral waves, ψ^- vanishes.

4 Conclusion

Finally, let us compare the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ and the $(1,0)\oplus(0,1)$ case side by side:

	Dirac spinor Ψ	**EM tensor** $F^{\mu\nu}$
Representation of Lorentz group	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$(1,0)\oplus(0,1)$
Left-chiral component	$(\frac{1}{2},0)$: Weyl spinor ψ_L	$(1,0)$: RS-vector ψ^+
Right-chiral component	$(0,\frac{1}{2})$: Weyl spinor ψ_R	$(0,1)$: RS-vector ψ^-
Equations of motion	$\mathrm{i}\frac{\partial\psi_{R,L}}{\partial t} = \pm(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_{R,L}$	$\mathrm{i}\frac{\partial\boldsymbol{\psi}^\pm}{\partial t} = \pm(\boldsymbol{\Sigma}\cdot\mathbf{p})\boldsymbol{\psi}^\pm$

Table 1: Summary of our results.

For the spin-$\frac{1}{2}$ case, we were able to express the four-component Dirac spinor Ψ via two two-component Weyl spinors $\psi_{R,L}$. The Weyl spinors have a well-defined chirality and obey different equations of motion.

In the spin-1 case, we have the option to describe the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ via six components of the electromagnetic field strength tensor $F^{\mu\nu}$, or via two three-component Riemann–Silberstein vectors $\psi^\pm$. Like the Weyl spinors, they have well-defined chiralities and follow different equations of motion.

References

[1] M. Nakahara, *Geometry, Topology and Physics.* CRC Press, 2003.

[4] These are the spin-1 generators for the group $SU(2)$, in other words the generators in the adjoint representation.