Quantum Effective Action

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Abstract

We review the quantum effective action Γ in a quantum field theory. We introduce Γ as the Legendre transformation of the generating functional of connected Green's functions and show some interesting properties in Section 2. Section 3 shows how Γ is connected to the ground state energy of a system. Section 4 introduces the loop expansion of the effective action and Section 5 applies this to a φ^4 theory.

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1 Pedagogical Approach

Following Ref. [1], we want to find an effective action Γ, so that working with this new action at tree-level only is equivalent to the whole theory using the classical action S. This is only possible if Γ contains proper (1PI) vertices,

$$\mathrm{i}\Gamma[\varphi_J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1^\mu} \cdots \int_{x_n^\mu} \boldsymbol{V}_n(x_1, \cdots, x_n) \varphi_J(x_1) \cdots \varphi_J(x_n). \tag{1}$$

To find out, how Γ is related to quantities that we already know, consider the path integral, where we switch from S to Γ:

$$\mathcal{Z}[J] \equiv \tfrac{1}{Z[0]} \int \mathcal{D}\varphi \, \mathrm{e}^{\mathrm{i}S[\varphi]+\mathrm{i}\int_{x^\mu} J\varphi} \equiv \mathrm{e}^{\mathrm{i}W[J]} \quad \to \quad \mathcal{Z}_\Gamma[J] \equiv \tfrac{1}{Z_\Gamma[0]} \int \mathcal{D}\varphi \, \mathrm{e}^{\mathrm{i}\Gamma[\varphi]+\mathrm{i}\int_{x^\mu} J\varphi} \equiv \mathrm{e}^{\mathrm{i}W_\Gamma[J]} \tag{2}$$

The defining property of Γ is that using it at tree-level is equivalent to using S at all loop orders. So in order to isolate the tree-level from this expression, we use the following trick. We re-introduce factors of $\hbar$ to the expression:

$$\mathcal{Z}_\Gamma[J] \equiv \tfrac{1}{Z_\Gamma[0]} \int \mathcal{D}\varphi \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Gamma[\varphi]+\frac{\mathrm{i}}{\hbar}\int_{x^\mu} J\varphi} \tag{3}$$

How will a Feynman diagram depend on $\hbar$? Inside such a diagram, we have propagators, vertices and sources. We will later prove that $\boldsymbol{V}_2$ is an inverse propagator and $\boldsymbol{V}_{n\geq 3}$ are proper Green's functions. This means, for P propagator lines, the diagram will have a factor $\hbar^P$, and for V vertices, there will be a factor $\hbar^{-V}$. And J source terms will yield a $\hbar^{-J}$. In total, we have:

$$\text{diagram} \propto \hbar^{P-V-J}. \tag{4}$$

We can express $P - J$ as I, the number of *internal* lines. This means the power is $\hbar^{I-V}$. Additionally, we can use the general statement for a connected graph that $V - I + L = 1$, therefore the power of $\hbar$ is

$$\text{diagram} \propto \hbar^{L-1}. \tag{5}$$

We can write W_Γ in a series by counting loops, knowing that $W_\Gamma[J]\Big|_{L=0} \overset{\text{def}}{=} W[J]$

$$\frac{1}{\hbar} W_\Gamma[J] = \sum_{L=0}^{\infty} \hbar^{L-1} W_\Gamma^{(L)}[J] \quad \to \quad \lim_{\hbar\to 0} W_\Gamma[J] = W_\Gamma^{(0)}[J] + \text{negligible} \tag{6}$$

The introduction of $\hbar$ gave us a simple tool to single out the tree diagrams: taking the limit $\hbar \to 0$. If we now look at eq. (3) in this limit, we see that the action in the path integral becomes very large. In such a case, one can show[1] that the path integral behaves like

$$\lim_{\hbar\to 0} \mathcal{Z}_\Gamma[J] \sim \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Gamma[\varphi_J]+\frac{\mathrm{i}}{\hbar}\int_{x^\mu} J\varphi_J}, \quad \text{where } \frac{\delta(\Gamma + \int J\varphi)}{\delta\varphi}\Big|_{\varphi=\varphi_J} = 0. \tag{7}$$

$\varphi_J(x; J)$ is a field configuration[2] where this new action Γ is extremal. We can now express both $\mathcal{Z}_\Gamma$ and W_Γ in eq. (2) in the limit of small $\hbar$ using eqs. (6) and (7), respectively. Comparing the leading terms yields the equality,

$$\Gamma[\varphi_J] + \int_{x^\mu} J(x)\varphi_J(x) = \hbar^{-1} W_\Gamma^{(0)}[J] \equiv W[J], \tag{8}$$

which tells us how to calculate Γ in terms of W: by doing a Legendre transformation.

[1] Keyword: *method of steepest descent.*

[2] The index J reminds us of the dependence on the source $J(x)$.

2 Systematic Approach

The quantum effective action is defined as the Legendre transformation of the generating functional $W[J]$ for connected Green's functions [2]. To do this Legendre transformation, we need the first derivative of W,

$$\frac{\delta W[J]}{\delta J(x)} = \frac{\delta}{\mathrm{i}\delta J(x)} \ln Z[J] = \frac{1}{Z[J]} \frac{\delta Z[J]}{\mathrm{i}\delta J(x)} = \langle \Omega | \mathsf{T} \varphi(x) | \Omega \rangle_J \equiv \varphi_J(x) \tag{9}$$

Here we defined the field $\varphi_J(x)$. The index J reminds us that it depends on the source. $\varphi_J(x)$ is not a quantum field, but a regular c-number valued field, because it is an expectation value of the field $\varphi(x)$ *in the presence* of the source $J(x)$. The index J after the vacuum expectation value signifies that we do *not* set the source to zero. After this initial setup we can perform the Legendre transformation:

$$\Gamma[\varphi_J] = W[J] - \int\limits_{y^\mu} J(y)\varphi_J(y) \bigg|_{J(x)\to J(x;\varphi_J)} \tag{10}$$

The left-hand side signifies that the whole expression only depends on φ_J, not on the current anymore. Therefore we have to replace the current J with a function that depends on φ_J. We get this function by inverting the relation eq. (9), since the left-hand side is a functional of J. Now we are ready to prove several things:

1. The first derivative of Γ: Let us take the derivative of the effective action with respect to the field φ_J:

$$\begin{aligned} \frac{\delta \Gamma[\varphi_J]}{\delta \varphi_J(x)} &= \frac{\delta W[J]}{\delta \varphi_J(x)} - \frac{\delta}{\delta \varphi_J(x)} \int\limits_{y^\mu} J(y)\varphi_J(y) \\ &= \int\limits_{y^\mu} \frac{\delta J(y)}{\delta \varphi_J(x)} \frac{\delta W[J]}{\delta J(y)} - \int\limits_{y^\mu} \frac{\delta J(y)}{\delta \varphi_J(x)} \varphi_J(y) - J(x) \end{aligned} \tag{11}$$

Due to eq. (9), the first two terms cancel and we see that the first derivative of Γ yields the current:

$$\frac{\delta \Gamma[\varphi_J]}{\delta \varphi_J(x)} = -J(x) \tag{12}$$

2. Further derivatives of Γ: Using functional derivatives, we get proper n-point Green's functions, $\boldsymbol{V}_n$, also called proper vertices, from the quantum effective action,

$$\boldsymbol{V}_n(x_1, \cdots, x_n) = \frac{\delta^n \mathrm{i}\Gamma[\varphi_J]}{\delta \varphi_J(x_1) \cdots \varphi_J(x_n)} \tag{13}$$

A *proper* vertex $\boldsymbol{V}_n$ is defined by two properties. First, it is a truncated Green's function and second, it belongs to an one-particle irreducible (1PI) graph. Let us prove this:

a) Generally, a truncated (or amputated) Green's function can be defined via

$$g_n(x_1, \cdots, x_n) = \int\limits_{x_1^\mu} \cdots \int\limits_{x_n^\mu} g_2(x_1, y_1) \cdots g_2(x_n, y_n) \tilde{g}_n^{\text{trunc.}}(y_1, \cdots y_n) \tag{14}$$

Graphically, this means cutting a dressed propagator from each external arm of the to-be truncated Green's function. Let us show explicitly that the 2- and 3-point proper vertices $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$ are in fact truncated Green's functions:

i) We start by replacing the current $J(z)$ in the following expression using eq. (12):

$$\delta^{(4)}(x-z) = \frac{\delta J(z)}{\delta J(x)} = -\frac{\delta^2 \Gamma[\varphi_J]}{\delta J(x)\delta\varphi_J(z)} \tag{15}$$

Next we have to use the chain rule because Γ does not explicitly depend on J:

$$\delta^{(4)}(x-z) = -\int_{u^\mu} \frac{\delta\varphi_J(u)}{\delta J(x)} \frac{\delta^2 \Gamma[\varphi_J]}{\delta\varphi_J(u)\delta\varphi_J(z)} \tag{16}$$

We substitute $\varphi_J(u)$ in the first term using eq. (9) and add some factors of i so that we can replace the derivatives of the generating functionals with corresponding Green's functions,

$$\delta^{(4)}(x-z) = -\int_{u^\mu} \frac{\delta \mathrm{i} W[J]}{\mathrm{i}\delta J(x)\mathrm{i}\delta J(u)} \frac{\delta^2 \mathrm{i}\Gamma[\varphi_J]}{\delta\varphi_J(u)\delta\varphi_J(z)} \tag{17}$$

Setting the source to zero, we can express this in Green's functions,

$$\delta^{(4)}(x-z) = -\int_{u^\mu} g_2(x,u)\boldsymbol{V}_2(u,z) \tag{18}$$

Finally, we can do a convolution with $g_2(z,y)$ and integrate over z^μ:

$$g_2(x,y) = -\int_{u^\mu z^\mu} g_2(x,u) g_2(y,z)\boldsymbol{V}_2(u,z) \tag{19}$$

and this is exactly the relation for a truncated Green's function, eq. (14). As a collorary, we see that

$$\boldsymbol{V}_2(x,y) = -[g_2(x,y)]^{-1} \tag{20}$$

ii) We start with the definition of the 3-point connected Green's function,

$$g_3(x,y,z) = \frac{\delta}{\mathrm{i}\delta J(x)} \frac{\delta^2 \mathrm{i} W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(z)} \tag{21}$$

where we isolated the derivative with respect to $J(x)$. Next we use a matrix identity:

$$\mathrm{d}\left(M^{-1}_{ij}\right) = -M^{-1}_{im}(\mathrm{d}M_{mn})M^{-1}_{nj}, \qquad M^{-1}_{yz} \overset{\text{here}}{=} \frac{\delta^2 \mathrm{i} W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(z)} \tag{22}$$

which we prove in the appendix. So we have

$$\frac{\delta}{\mathrm{i}\delta J(x)} \frac{\delta^2 \mathrm{i} W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(z)} = -\int_{a^\mu b^\mu} \frac{\delta^2 \mathrm{i} W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(a)} \left(-\frac{\delta}{\mathrm{i}\delta J(x)} \frac{\delta^2 \mathrm{i}\Gamma[\varphi_J]}{\delta\varphi_J(a)\delta\varphi_J(b)}\right) \frac{\delta^2 \mathrm{i} W[J]}{\mathrm{i}\delta J(b)\mathrm{i}\delta J(z)} \tag{23}$$

We use the chain rule to add a $\varphi_J(c)$,

$$\frac{\delta}{\mathrm{i}\delta J(x)}\frac{\delta^2 \mathrm{i}W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(z)} = \int\limits_{a^\mu b^\mu c^\mu} \frac{\delta^2 \mathrm{i}W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(a)}\left(\frac{\delta\varphi_J(c)}{\mathrm{i}\delta J(x)}\frac{\delta^2\mathrm{i}\Gamma[\varphi_J]}{\delta\varphi_J(a)\delta\varphi_J(b)\delta\varphi_J(c)}\right)\frac{\delta^2\mathrm{i}W[J]}{\mathrm{i}\delta J(b)\mathrm{i}\delta J(z)} \tag{24}$$

and use eq. (9) to replace the $\varphi_J(c)$ in the first term in brackets,

$$\frac{\delta}{\mathrm{i}\delta J(x)}\frac{\delta^2 \mathrm{i}W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(z)} = \int\limits_{a^\mu b^\mu c^\mu} \frac{\delta^2 \mathrm{i}W[J]}{\mathrm{i}\delta J(y)\mathrm{i}\delta J(a)}\left(\frac{\delta\mathrm{i}W[J]}{\mathrm{i}\delta J(x)\mathrm{i}\delta J(c)}\frac{\delta^2\mathrm{i}\Gamma[\varphi_J]}{\delta\varphi_J(a)\delta\varphi_J(b)\delta\varphi_J(c)}\right)\frac{\delta^2\mathrm{i}W[J]}{\mathrm{i}\delta J(b)\mathrm{i}\delta J(z)}$$

Now we can finally set the sources to zero and write the equation in terms of Green's functions:

$$g_3(x,y,z) = \int\limits_{a^\mu b^\mu c^\mu} g_2(y,a)g_2(x,c)\boldsymbol{V}_3(a,b,c)g_2(b,z) \tag{25}$$

and again, this is analogous to the defining equation for a truncated Green's function, eq. (14).

iii) In the same way, one can show that all other proper vertices $\boldsymbol{V}_n$ for $n \geq 3$ are truncated Green's functions.

b) One-particle irreducible (1PI) means that we cannot separate the graph for a Green's function by cutting only one line. To start the proof that $\boldsymbol{V}_2$ is a 1PI Green's function, we consider g_2. Since this corresponds to connected Green's functions, we can classify them in the following way. The first contribution to g_2 is the free propagator Δ,

$$g_2 = \mathrm{i}\Delta + \cdots\,, \qquad \Delta(p) = \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon} \tag{26}$$

The next term should have two external legs, and within these external lines all sorts of interactions may take place. We sort them by saying that in the next terms, we want to have an increasing number of 1PI diagrams, which we will denote with $\mathrm{i}\Sigma$,

$$g_2 = \mathrm{i}\Delta + \mathrm{i}\Delta(\mathrm{i}\Sigma)\mathrm{i}\Delta + \mathrm{i}\Delta(\mathrm{i}\Sigma)\mathrm{i}\Delta(\mathrm{i}\Sigma)\mathrm{i}\Delta + \cdots \tag{27}$$

Next we write this into a series,

$$g_2 = \mathrm{i}\Delta\Big[1 + (\mathrm{i}\Sigma)\mathrm{i}\Delta + (\mathrm{i}\Sigma)\mathrm{i}\Delta(\mathrm{i}\Sigma)\mathrm{i}\Delta + \cdots\Big] \tag{28a}$$

$$= \mathrm{i}\Delta\Big[1 + (\mathrm{i}\Sigma)\mathrm{i}\Delta + [(\mathrm{i}\Sigma)\mathrm{i}\Delta]^2 + \cdots\Big] \tag{28b}$$

$$= \frac{\mathrm{i}\Delta}{1 - \mathrm{i}\Sigma\mathrm{i}\Delta} \qquad \text{using } \sum_n q^n = \frac{1}{1-q} \tag{28c}$$

$$= \frac{\mathrm{i}}{\Delta^{-1} + \Sigma} \tag{28d}$$

The inverse of this relation is

$$g_2^{-1} = \frac{1}{\mathrm{i}}\Delta^{-1} + \frac{1}{\mathrm{i}}\Sigma \tag{29}$$

Next we replace g_2^{-1} with $\boldsymbol{V}_2$, as seen in eq. (20),

$$-\boldsymbol{V}_2 = \frac{1}{\mathrm{i}}\Delta^{-1} + \frac{1}{\mathrm{i}}\Sigma \tag{30}$$

and multiplication with i^2 shows us that $\boldsymbol{V}_2$ is really given only by 1PI expressions:

$$\boldsymbol{V}_2 = \mathrm{i}\Delta^{-1} + \mathrm{i}\Sigma \tag{31}$$

In the next sections, we will look at some more properties of the quantum effective action.

3 Intepretation as Ground State Energy

It is sometimes useful to express the quantum effective action $\Gamma[\varphi_J]$ in a derivative series,

$$\Gamma[\varphi_J] = \int \mathrm{d}^4x \left[-\mathcal{U}(\varphi_J) + \frac{1}{2}\mathbf{Z}(\varphi_J)\partial_\mu\varphi_J\partial^\mu\varphi_J + \mathbf{Y}(\varphi_J)\partial_\mu\varphi_J\partial^\mu\varphi_J\partial_\nu\varphi_J\partial^\nu\varphi_J + \cdots\right]. \tag{32}$$

In the following we will focus on the first term in this expansion and investigate the physical meaning of $\mathcal{U}(\varphi_J)$, following Ref. [1].

Suppose that at time $t = -\infty$, we turn on a source $J(t)$ very slowly. This source reaches its highest value $J(0) = \hat{J}$ at time $t = 0$ and stays like this for a time interval $\Delta t = T$. After that, we slowly turn off this source, such that $J(\infty) = 0$ again. If we start with the vacuum state $|\Omega\rangle$, we will be in the vacuum state again. However, if we consider the following fraction, we will pick up these phase factors:

$$\frac{\langle\Omega|\Omega\rangle_J}{\langle\Omega|\Omega\rangle} = \frac{\mathrm{e}^{-\mathrm{i}E_g^J T}}{\mathrm{e}^{-\mathrm{i}E_\Omega T}}, \qquad \text{with } E_g^J = \langle g|H_J|g\rangle\,, \quad E_\Omega = \langle\Omega|H|\Omega\rangle \tag{33}$$

Since the activation of the source happened very slowly (adiabatically), we can assume that we were again in the ground state between $t = 0$ and $t = T$, which we denoted with $|g\rangle$. Let us now shift our energy scale such that $E_\Omega = 0$. Since the fraction in eq. (33) is exactly the generating functional for full Green's functions, we can write

$$\mathcal{Z}[J] = \mathrm{e}^{-\mathrm{i}E_g^J T} = \mathrm{e}^{\mathrm{i}W[J]} \quad \leftrightarrow \quad -\frac{W[J]}{T} = \langle g|H_J|g\rangle\,. \tag{34}$$

So we see that the negative energy of the ground state in the presence of $\hat{J}$, E_g^J, is given by the generating functional of connected Green's functions, divided by the time during which the source was active. Now consider the Hamiltonian H_J is given by,

$$H_J = H - \int_{\boldsymbol{x}} J\varphi \tag{35}$$

We can take the expectation value of this equation in the state $|g\rangle$ and replace the left-hand side using eq. (34) and multiply the time T to the right-hand side, which turns the three-dimensional integral into a four-dimensional integral,

$$\langle g|H_J|g\rangle = \langle g|H|g\rangle - \int_{\boldsymbol{x}} J\,\langle g|\varphi|g\rangle \tag{36a}$$

$$-W[J] = \langle g|H|g\rangle\, T - \int_{x^\mu} J\,\langle g|\varphi|g\rangle \tag{36b}$$

We use the Legendre transformation, eq. (10), and under the assumption that φ_J does not depend on spacetime ($\partial_\mu \varphi_J = 0$), we arrive at

$$\Gamma[\varphi_J] \overset{(36b)}{=} -\int_{x^\mu} \langle g|\mathcal{H}|g\rangle \overset{(32)}{=} -\int_{x^\mu} \mathcal{U}(\varphi_J). \tag{37}$$

Therefore, $\mathcal{U}$ is the energy density of the ground state. We can even go a step further: Remembering the first derivative of the effective action, eq. (12), setting the sources to zero and demanding that φ_J is constant in spacetime, this equation simplifies to

$$-\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J} \overset{(32)}{=} \frac{\delta\mathcal{U}(\varphi_J)}{\delta\varphi_J} = 0. \tag{38}$$

This equation can be used to determine the ground state φ_J, therefore we should interpret $\mathcal{U}$ as a potential. We call it the "quantum effective potential".

4 Loop Expansion

In the last section we got to know the derivative expansion of the effective action, eq. (32). In this section, we will investigate how the effective potential $\mathcal{U}$ looks like. This can be done step by step, every time considering more loops. Therefore this process is called *loop expansion* [3]. We start with the path integral, which can be evaluated approximately,

$$\mathcal{Z}[J] = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}W[J]} = \tfrac{1}{Z[0]}\int \mathcal{D}\varphi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left[S[\varphi]+\int J\varphi\right]} \approx \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left[S[\varphi_0]+\int J\varphi_0\right]}, \quad \frac{\delta\left[S[\varphi]+\int J\varphi\right]}{\delta\varphi}\Big|_{\varphi=\varphi_0} = 0 \tag{39}$$

if φ_0 corresponds to an extremal value of the action[3]. For a Lagrangian of a real scalar field, this condition can be explicitly stated as $-\partial_\mu\partial^\mu\varphi_0 - V'[\varphi_0] + J = 0$.

Let us explicitly calculate what we stated in eq. (39). We claim that the action is stationary for $\varphi = \varphi_0$, i.e. the first derivative of [action plus source term] vanishes as seen in eq. (39). Let $\varphi = \varphi_0 + \tilde{\varphi}$ and expand the action:

$$S[\varphi] = S[\varphi_0] + \frac{\delta S[\varphi]}{\delta\varphi}\Big|_{\varphi=\varphi_0}\tilde{\varphi} + \frac{1}{2}\frac{\delta^2 S[\varphi]}{\delta\varphi^2}\Big|_{\varphi=\varphi_0}\tilde{\varphi}^2 + \cdots . \tag{40}$$

Ignoring the dots, the generating functional is given by

$$\mathcal{Z}[J] \approx \tfrac{1}{Z[0]}\int \mathcal{D}\tilde{\varphi}\, \exp\Big(\tfrac{\mathrm{i}}{\hbar}\Big[S[\varphi_0] + \tfrac{\delta S[\varphi]}{\delta\varphi}\Big|_{\varphi=\varphi_0}\tilde{\varphi} + \tfrac{1}{2}\tfrac{\delta^2 S[\varphi]}{\delta\varphi^2}\Big|_{\varphi=\varphi_0}\tilde{\varphi}^2 + \int J(\varphi_0+\tilde{\varphi})\Big]\Big) \tag{41a}$$

$$= \tfrac{1}{Z[0]}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\varphi_0]}\int \mathcal{D}\tilde{\varphi}\, \exp\Big(\tfrac{\mathrm{i}}{\hbar}\Big[\int(-J)\tilde{\varphi} + \tfrac{1}{2}\tfrac{\delta^2 S[\varphi]}{\delta\varphi^2}\Big|_{\varphi=\varphi_0}\tilde{\varphi}^2 + \int J(\varphi_0+\tilde{\varphi})\Big]\Big) \tag{41b}$$

$$= \tfrac{1}{Z[0]}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S[\varphi_0]+\int J\varphi_0)}\int \mathcal{D}\tilde{\varphi}\, \exp\Big(\tfrac{\mathrm{i}}{\hbar}\Big[\tfrac{1}{2}\tfrac{\delta^2 S[\varphi]}{\delta\varphi^2}\Big|_{\varphi=\varphi_0}\tilde{\varphi}^2\Big]\Big) \tag{41c}$$

$$= \tfrac{1}{Z[0]}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S[\varphi_0]+\int J\varphi_0)}\mathcal{N}\,\mathrm{Det}(\partial^2 + V''[\varphi_0])^{-1/2}, \tag{41d}$$

where $\mathcal{N}$ is a diverging constant. We use the identity "$\ln\det = \mathrm{tr}\log$" (see appendix),

$$\mathcal{Z}[J] = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}W[J]} = \tfrac{\mathcal{N}}{Z[0]}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S[\varphi_0]+\int J\varphi_0) - \frac{1}{2}\mathrm{Tr}\log\left(\partial^2+V''[\varphi_0]\right)} \tag{42}$$

[3]Keyword: *method of steepest descent*.

or equivalently (we will ignore the diverging constant after this equation),

$$W[J] = S[\varphi_0] + \int J\varphi_0 + \frac{\mathrm{i}\hbar}{2}\operatorname{Tr}\log\Big(\partial^2 + V''[\varphi_0]\Big) + \log\Big(\mathcal{N}Z[0]^{-1}\Big) \tag{43}$$

From this equation we can see that[4] $\varphi_J \equiv \frac{\delta W[J]}{\delta J} = \varphi_0 + \mathcal{O}(\hbar)$, so up to leading order in $\hbar$, we can replace φ_0 by φ_J, which allows us to do the Legendre transformation eq. (10):

$$\Gamma[\varphi_J] = S[\varphi_J] + \frac{\mathrm{i}\hbar}{2}\operatorname{Tr}\log\Big(\partial^2 + V''[\varphi_J]\Big) + \mathcal{O}(\hbar^2) \tag{44}$$

In practice it is impossible to evaluate this expression for arbitrary fields. We would have to find all eigenvalues of the operator, take their logarithm, and sum them. However, for special φ_J, which is independent of x, we can continue the calculation of the trace-log. First, we insert a completeness relation in momentum space:

$$\operatorname{Tr}\log\Big(\partial^2 + V''[\varphi_J]\Big) = \int\limits_{x^\mu}\int \mathrm{d}^4p\,\langle x|\log\Big(\partial^2 + V''[\varphi_J]\Big)|p\rangle\langle p|x\rangle \tag{45a}$$

$$= \int\limits_{x^\mu}\int \mathrm{d}^4p\,\langle x|p\rangle\log\Big(-p^2 + V''[\varphi_J]\Big)\langle p|x\rangle \tag{45b}$$

We use eq. (32) to get an expression for the Coleman–Weinberg effective potential:

$$\mathcal{U}(\varphi_J) = V[\varphi_J] - \frac{\mathrm{i}\hbar}{2}\int\limits_{p^\mu}\log\left(\frac{-p^2 + V''[\varphi_J]}{-p^2}\right) + \mathcal{O}(\hbar^2). \tag{46}$$

$$= \mathcal{U}^{(0)}(\varphi_J) + \mathcal{U}^{(1)}(\varphi_J) + \mathcal{O}(\hbar^2)$$

This is the first $\mathcal{O}(\hbar)$ term for the quantum effective potential, where we added a φ_J-independent constant to have a dimensionless argument in the logarithm. For the one-loop correction, we need to calculate the following integral,

$$\mathcal{U}^{(1)}(\varphi_J) = -\frac{\mathrm{i}\hbar}{2}\int\limits_{p^\mu}\log\Big(\tfrac{-p^2+V''[\varphi_J]}{-p^2}\Big) \tag{47}$$

We substitute $p^0 = \mathrm{i}p^4$ and notice that the integrand only contains the momentum squared so we can simplify (proof in the appendix),

$$\mathcal{U}^{(1)}(\varphi_J) = \frac{\hbar}{2(2\pi)^4}\int \mathrm{d}^4p_{\scriptscriptstyle E}\log\left(\frac{p_{\scriptscriptstyle E}^2 + V''}{p_{\scriptscriptstyle E}^2}\right), \quad \mathrm{d}^4p_{\scriptscriptstyle E} = \mathrm{d}p^1\mathrm{d}p^2\mathrm{d}p^3\mathrm{d}p^4 \tag{48a}$$

$$= \frac{\hbar}{32\pi^2}\int_0^\infty \mathrm{d}\xi\,\xi\log\left(\frac{\xi + V''}{\xi}\right), \quad \text{since } \int \mathrm{d}^4p_{\scriptscriptstyle E} f(p_{\scriptscriptstyle E}^2) = \pi^2\int_0^\infty \mathrm{d}\xi\;\xi f(\xi) \tag{48b}$$

We regularize this integral by only integrating up to Λ^2, instead of infinity, which yields

$$\int_0^\infty \mathrm{d}\xi\,\xi\log\left(\frac{\xi + V''}{\xi}\right) = \frac{\Lambda^2}{2}V'' + \frac{(V'')^2}{2}\log\left(\frac{V''}{\Lambda^2 + V''}\right) + \frac{\Lambda^4}{2}\log\left(\frac{V'' + \Lambda^2}{\Lambda^2}\right) \tag{49}$$

In the limit $\Lambda \to \infty$, the only non-zero terms are,

$$\lim_{\Lambda\to\infty}\int_0^\infty \mathrm{d}\xi\,\xi\log\left(\frac{\xi + V''}{\xi}\right) = \Lambda^2 V'' + \frac{(V'')^2}{2}\log\left(\frac{V''}{\Lambda^2\,\mathrm{e}^{1/2}}\right) \tag{50}$$

Therefore, the one-loop effective potential is,

$$\mathcal{U}^{(1)}(\varphi_J) = \frac{\hbar}{32\pi^2}\Lambda^2 V'' + \frac{\hbar(V'')^2}{64\pi^2}\log\left(\frac{V''}{\Lambda^2\,\mathrm{e}^{1/2}}\right) \tag{51}$$

[4] Remember that φ_0 depends on J and also use eq. (39).

5 Effective Potential: Concrete Example

We consider a renormalized φ^4 theory for a real scalar field. This means, we are looking at the bare Lagrangian,

$$\mathcal{L}_{\text{bare}} = \frac{1}{2}\partial_\mu\varphi_{(0)}\partial^\mu\varphi_{(0)} - \frac{m^2_{(0)}}{2}\varphi^2_{(0)} - \frac{\lambda_{(0)}}{4!}\varphi^4_{(0)}. \tag{52}$$

This Lagrangian contains bare quantities, which are allowed to diverge, so we would rather work with the Lagrangian containing the physical fields. It is given in terms of renormalization coefficients Z,

$$\mathcal{L} = \tfrac{1}{2}Z_\varphi\partial_\mu\varphi\partial^\mu\varphi - Z_m\tfrac{m^2}{2}\varphi^2 - Z_\lambda\tfrac{\lambda}{4!}\varphi^4. \tag{53}$$

The bare quantities and the renormalized ones are related via,

$$\sqrt{Z_\varphi}\,\varphi = \varphi_{(0)}, \quad Z_m m\varphi^2 = \tilde{Z}_m m Z_\varphi\varphi^2 = m_{(0)}\varphi^2_{(0)}, \quad Z_\lambda\lambda\varphi^4 = \tilde{Z}_\lambda\lambda Z^2_\varphi\varphi^4 = \lambda_{(0)}\varphi^4_{(0)} \tag{54}$$

To lowest order, the renormalization coefficients are one, therefore we introduce new parameters, $Z_\varphi = 1 + A$, $Z_m = 1 + B$ and $Z_\lambda = 1 + C$. Now the Lagrangian looks like the bare one but contains physical fields and so-called counterterms,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}A(\partial_\mu\varphi)^2 - \frac{m}{2}B\varphi^2 - \frac{\lambda}{4!}C\varphi^4. \tag{55}$$

In the last section, we calculated the quantum effective potential in the loop expansion in eq. (46). Since the counterterms are of order $\mathcal{O}(\hbar)$,

$$A = \hbar A_1 + \hbar^2 A_2 + \cdots \qquad B = \hbar B_1 + \hbar^2 B_2 + \cdots \qquad C = \hbar C_1 + \hbar^2 C_2 + \cdots \tag{56}$$

the effective potential is given in tree approximation as

$$\mathcal{U}(\varphi_J) = \mathcal{U}^{(0)}(\varphi_J) + \mathcal{O}(\hbar) = V[\varphi_J] + \mathcal{O}(\hbar) = \tfrac{m}{2}\varphi_J^2 + \tfrac{\lambda}{4!}\varphi_J^4 + \mathcal{O}(\hbar). \tag{57}$$

The one-loop term can be evaluated using eq. (51),

$$\mathcal{U}^{(1)}(\varphi_J) = \frac{\hbar\Lambda^2}{32\pi^2}(m^2 + \frac{\lambda}{2}\varphi_J^2) + \frac{\hbar(m^2 + \frac{\lambda}{2}\varphi_J^2)^2}{64\pi^2}\log\left(\frac{m^2 + \frac{\lambda}{2}\varphi_J^2}{\Lambda^2\,\mathrm{e}^{1/2}}\right) + \frac{\hbar m}{2}B_1\varphi_J^2 + \frac{\hbar\lambda}{4!}C_1\varphi_J^4. \tag{58}$$

This equation contains two diverging terms as $\Lambda \to \infty$, namely the first ("quadratic divergence") and second one ("logarithmic divergence"). This is where the counterterms become important. The diverging terms contain either a φ_J^2 or a φ_J^4, therefore B_1 and C_1 can be used to absorb these divergences, such that the effective potential $\mathcal{U}$ is still well-defined.

This worked, because initially, V was a polynomial of degree 4 in the fields, which produces counterterms up to degree 4. Then, V'' is of degree 2 and $(V'')^2$ is of degree 4. Therefore the counterterms match the divergences. For an example of a non-renormalizable theory, consider a potential of degree 6. This yields counterterms up to sixth order, but $(V'')^2$ is of order 8. So we do not have a counterterm that could absorb a φ^8 divergence. We could start with a potential of order 8, but then we would need even more counterterms. This need for infinitely many counterterms is a general property of a non-renormalizable theory.

A Appendix

A.1 Derivative of an Inverse Matrix

Since the derivative of a constant is zero,

$$\mathrm{d}\mathbb{1} = 0, \tag{59}$$

and the inverse fulfills $MM^{-1} = \mathbb{1}$, we can write this as

$$\mathrm{d}\Big(MM^{-1}\Big) = 0. \tag{60}$$

Applying the product rule yields,

$$\mathrm{d}MM^{-1} + M\mathrm{d}M^{-1} = 0, \tag{61}$$

and rearranging for $\mathrm{d}M^{-1}$ gives the result:

$$\mathrm{d}M^{-1} = -M^{-1}\mathrm{d}MM^{-1} \tag{62}$$

A.2 Higher Dimensional Integrals

In an n-dimensional Euclidean space, the integral over all n coordinates yields the n-dimensional ball of radius R, $V_n(R)$, which is a well-known result,

$$V_n(R) = \int_{|x|\leq R} \mathrm{d}^n x = V_n R^n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^n, \qquad \Gamma(n) = (n-1)! \tag{63}$$

For a spherically symmetrical integrand, this integral can be equivalently written as a one-dimensional integral over a generalized radius ρ, using the surface of the n-ball, the so-called $(n-1)$-sphere, $S_{n-1}(R)$,

$$V_n(R) = \int_0^R \mathrm{d}\rho\, S_{n-1}(\rho) = \int_0^R \mathrm{d}\rho\, S_{n-1}\rho^{n-1}, \qquad S_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \tag{64}$$

The first few values for the n-ball and the n-sphere are discussed in the following items. Note that the volume always has one dimension more than the surface!

Zero dimensions: The 0-ball, V_0, is a point, therefore $V_0(R) = 1$. This point does not have a surface.

One dimension: The 1-ball, V_1, is the interval $[-R, R]$, therefore $V_2(R) = 2R$. Its surface is the zero-sphere, S_0, which consists of the two points $\{-R, R\}$. Therefore, $S_0(R) = 2$.

Two dimensions: The 2-ball is the area of a circle of radius R, so $V_2(R) = \pi R^2$. Its boundary is the 1-sphere, S_1, which is a circle of radius R, so $S_1(R) = 2\pi R$.

Three dimensions: The 3-ball is the volume of a sphere of radius R, so $V_3(R) = \frac{4\pi}{3}R^3$. Its boundary is the 2-sphere, S_2, which is the area $S_2(R) = 4\pi R^2$.

	0	**1**	**2**	**3**	**4**
V_n	1	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$
S_n	2	2π	4π	$2\pi^2$	$\frac{8\pi^2}{3}$

Table 1: The first five values for the n-ball and the n-sphere. Note that $V_n(R) = V_n R^n$ and $S_n(R) = S_n R^n$.

In particular, we use the four-dimensional case,

$$\int \mathrm{d}^4 p_{\scriptscriptstyle E} = \int_0^\infty \mathrm{d}\rho\, S_3 \rho^3 = 2\pi^2 \int_0^\infty \mathrm{d}\rho\, \rho^3 \tag{65}$$

and by substituting $\xi = \rho^2$, which means $\mathrm{d}\xi = 2\rho\mathrm{d}\rho$, we get the wanted expression for eq. (48b),

$$\int \mathrm{d}^4 p_{\scriptscriptstyle E} = \pi^2 \int_0^\infty \mathrm{d}\xi\, \xi, \tag{66}$$

in the case of a spherically symmetric integrand.

A.3 About the Classical Field

In classical mechanics, we determine the evolution of a system by looking for a stationary point of the action, i.e. we solve the equation,

$$\frac{\delta S[\varphi]}{\delta \varphi} = 0. \tag{67}$$

In eq. (7), we defined the field φ_J as the solution of a similar equation, this time with the quantum effective action,

$$\left.\frac{\delta \Gamma}{\delta \varphi(x)}\right|_{\varphi=\varphi_J} = -J(x). \tag{7}$$

We will now show that the solution to the quantum equations of motion, φ_J, is equal to the vacuum expectation value of the quantum field $\varphi(x)$ *in the presence of the sources*. We start with a functional derivative of the generating functional of connected Green's functions, $W[J]$, with respect to a source:

$$\frac{\delta W[J]}{\delta J(x)} \tag{68}$$

In eq. (8) we calculated how to express $W[J]$ in terms of $\Gamma[\varphi_J]$, so the functional derivative is given by,

$$\frac{\delta W[J]}{\delta J(x)} = \frac{\delta \Gamma[\varphi_J]}{\delta J(x)} + \varphi_J(x) + \int_{y^\mu} J(y) \frac{\delta \varphi_J(y)}{\delta J(x)}. \tag{69}$$

Since the effective action is not explicitly dependent on the sources, we use the chain rule for the first term,

$$\frac{\delta W[J]}{\delta J(x)} = \int_{y^\mu} \frac{\delta \Gamma[\varphi_J]}{\delta \varphi_J(y)} \frac{\delta \varphi_J(y)}{\delta J(x)} + \varphi_J(x) + \int_{y^\mu} J(y) \frac{\delta \varphi_J(y)}{\delta J(x)}, \tag{70}$$

and now we can combine the first and third term:

$$\frac{\delta W[J]}{\delta J(x)} = \int_{y^\mu} \left[\frac{\delta \Gamma[\varphi_J]}{\delta \varphi_J(y)} + J(y) \right] \frac{\delta \varphi_J(y)}{\delta J(x)} + \varphi_J(x) \tag{71}$$

The terms in brackets vanish, since they are exactly the quantum equations of motion. Therefore the solution to these quantum equations of motion, φ_J, is given by the vacuum expectation value of φ,

$$\varphi_J(x) = \frac{\delta W[J]}{\delta J(x)} = \langle \Omega | \mathsf{T} \varphi(x) | \Omega \rangle_J \,. \tag{72}$$

If we set the sources to zero, then this VEV vanishes[5].

A.4 log det = tr log

If a matrix M has eigenvalues λ_i, $\log(M)$ has eigenvalues $\log(\lambda_i)$. Then we can use basic properties of the logarithm function,

$$\mathrm{tr}(\log M) = \sum_i \log(\lambda_i) = \log\left(\prod_i \lambda_i \right) = \log(\det M) \tag{73}$$

which yields the desired result. NB: there might be ambiguities which branches of the logarithm to use when there are non-positive eigenvalues of M. A more precice statement would therefore be that $\mathrm{tr} \log(M)$ is one of the branches of $\log \det(M)$.

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[5] More precicely, we want it to vanish, in order to be able to use the LSZ reduction formula.