Currents and Symmetries

Stephan Hübsch*

March 2019

Abstract

Noether's first theorem connects global symmetries to conservation laws, whereas her second theorem is applicable for local symmetries and constrains the form of the equations of motion. In order to investigate these theorems, we need to compare two things: (i) "How does the action change under a general transformation?" and (ii) "How does the action change under a particular symmetry transformation?". These questions will be adressed in Sections 1 and 2. Consequently, Noether's first and second theorems are discussed in Sections 3 and 4.

Contents

*huebsch.s.aa@m.titech.ac.jp

1 General Transformations

We consider a Lagrangian density $\mathcal{L}$, which contains fields φ_k and their derivatives to N-th order. We will investigate what happens to the Lagrangian (and later the action) if we make a general transformation

$$\varphi_k(x) \to \varphi'_k(x) \overset{\text{infinitesimal}}{\approx} \varphi_k(x) + \delta\varphi_k(x) \tag{1}$$

The variation if the Lagrangian density is

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi_k}\delta\varphi_k + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_k)}\delta(\partial_\mu\varphi_k) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\varphi_k)}\delta(\partial_\mu\partial_\nu\varphi_k) + \dots \\ &= \frac{\partial\mathcal{L}}{\partial\varphi_k}\delta\varphi_k + \sum_{n=1}^{N}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}..\partial_{\mu_n}\varphi_k)}\delta(\partial_{\mu_1}..\partial_{\mu_n}\varphi_k)\end{aligned} \tag{2}$$

Note that we can freely switch the variation δ and the derivatives ∂_μ. After performing some partial integrations, we can separate these terms into two categories: terms that are multiplied by just $\delta\varphi$ (not its derivatives) and total derivatives. Here are the first two terms in the sum of eq. (2):

$$n=1 \qquad \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_k}\delta\partial_\mu\varphi_k = -\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_k}\delta\varphi_k + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_k}\delta\varphi_k\right] \tag{3a}$$

$$n=2 \quad \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k}\delta\partial_\mu\partial_\nu\varphi_k = \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k}\delta\varphi_k + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k}\partial_\nu\delta\varphi_k - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k}\delta\varphi_k\right] \tag{3b}$$

This shows that for *any* variation in the fields, the Lagrangian changes like

$$\delta\mathcal{L} = E^k\delta\varphi_k + \partial_\mu\theta^\mu, \tag{4}$$

where E^k represents the equations of motion (Euler-Lagrange equations) for the field φ_k and *all* other terms can be written as a total derivative of some function θ^μ. This θ^μ is a known function of the Lagrangian (see below) and often called "bare" Noether current.

The equations of motion (EoM) are defined as the functional derivatives of the action with respect to the fields. By comparing eqs. (2) to (4), we see

$$\begin{aligned}E^k &= \frac{\partial\mathcal{L}}{\partial\varphi_k} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_k} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k} \mp \dots \\ &= \frac{\partial\mathcal{L}}{\partial\varphi_k} + \sum_{n=1}^{N}(-1)^n\partial_{\mu_1}..\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}..\partial_{\mu_n}\varphi_k)}\end{aligned} \tag{5}$$

$$\begin{aligned}\theta^\mu &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_k}\delta\varphi_k + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k}\partial_\nu\delta\varphi_k - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi_k}\delta\varphi_k + \dots \\ &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_k}\delta\varphi_k + \sum_{n=1}^{N-1}\sum_{i=0}^{n}(-1)^i(\partial_{\mu_1}..\partial_{\mu_i})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\mu_1}..\partial_{\mu_n}\varphi_k)}(\partial_{\mu_1}..\partial_{\mu_{n-i}})\delta\varphi_k\end{aligned} \tag{6}$$

Note that the overall sign does not play a role in eqs. (5) and (6). In the next section, we will look at a specific transformation $\delta\varphi$.

2 Symmetry Transformations

We consider a specific transformation under a group G,

$$\hat{\delta}\varphi_k(x) = \mathrm{i}\omega_a(x)F_k^a[\varphi] + \mathrm{i}\partial_\mu\omega_a(x)F_k^{a\mu}[\varphi] \tag{7}$$

which can be parametrized by $a = 1, .., A$ infinitesimally small functions $\omega_a(x)$ and their derivatives to first order. The hat on the variation symbol signals that this is in fact a symmetry transformation, and therefore the Lagrangian can at most change by a total derivative. $F_k^{a(\mu)}$ are functionals of the fields and contain the generators of G.

If $\hat{\delta}\mathcal{L} = 0$, we speak of a strict symmetry, for $\hat{\delta}\mathcal{L} = \partial_\mu k^\mu$, we call $\hat{\delta}$ a quasi-symmetry. Since $\partial_\mu k^\mu$ is a boundary term, the action will not change in either case. Whether k^μ exists or not, depends on the particular transformation eq. (7).

3 Noether's First Theorem

For the first theorem, we consider global symmetry transformations. We have already seen in eq. (4) that under a general transformation, the Lagrangian changes as

$$\delta\mathcal{L} = E^k\delta\varphi_k + \partial_\mu\theta^\mu. \tag{4}$$

Now we will use a certain transformation δ, under which the Lagrangian changes like a total derivative plus another term that is due to explicit symmetry breaking:

$$\delta\mathcal{L} = E^k\delta\varphi_k + \partial_\mu\theta^\mu \overset{!}{=} \partial_\mu k^\mu + \mathsf{ESB} \tag{8}$$

We define the "full" Noether current $j^\mu \equiv \theta^\mu - k^\mu$, so that the divergence of this full current fulfills,

$$\partial_\mu j^\mu = \mathsf{ESB} - E^k\delta\varphi_k. \tag{9}$$

We conclude: in case that the equations of motion are fulfilled[1] and we have no explicit symmetry breaking, the full Noether current j^μ is conserved.

In the simpler case of $N = 1$, that is the Lagrangian only contains the fields and their first derivatives, the equations of motion and the full Noether current simplify to:

$$E^k = \frac{\partial\mathcal{L}}{\partial\varphi_k} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_k)} \qquad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_k)}\delta\varphi_k - k^\mu \tag{10}$$

In case we have a symmetry under several transformations, as parametrized by A parameters ω_a, we can identify A separate Noether currents and Noether charges:

$$j^{\mu a} = \frac{\partial j^\mu}{\partial\omega_a}, \qquad Q^a = \int \mathrm{d}^3x\, j^{0a}(x) \qquad (a = 1, .., A) \tag{11}$$

[1] This means $E^k = 0$ and is called "on-shell".

3.1 Current Algebra

In a quantum theory, the fields and their conjugate momenta obey equal-time (anti-) commutation relations,

$$[\varphi_k(t,\boldsymbol{x}), \pi_l(t,\boldsymbol{y})]_\pm = \mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{kl} \quad \pi_k \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0\varphi_k)}. \tag{12}$$

In the case of $N = 1$ and assuming a strict symmetry, the zero-th component of the full Noether current can be written as

$$j^0 = \pi_k \hat{\delta}\varphi_k = \omega_a j^{0a} \tag{13}$$

Since this expression is model-independent, i.e. it does not assume a specific shape of the Lagrangian, the following relations between the currents are also model-independent,

$$[j^{0a}(t,\boldsymbol{x}), j^{0b}(t,\boldsymbol{y})] = \mathrm{i}f^{abc} j^{0c}(t,\boldsymbol{x})\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{14}$$

which is called *current algebra*, since the currents fulfill the same algebra relations as the generators of the symmetry group G. After integrating over $\boldsymbol{x}$ and $\boldsymbol{y}$, one can see that the Noether charges fulfill these relations as well, which is refered to as *charge algebra*.

4 Noether's Second Theorem

For the second theorem, we consider local symmetry transformations. Again, we have seen in eq. (4) that under a general transformation, the Lagrangian changes as

$$\delta\mathcal{L} = E^k \delta\varphi_k + \partial_\mu \theta^\mu. \tag{4}$$

Let us consider the symmetry transformation in eq. (7). In case of a quasi-symmetry, the change in the Lagrangian can be equal to the correction term k^μ,

$$\hat{\delta}\mathcal{L} = E^k \hat{\delta}\varphi_k + \partial_\mu \theta^\mu \overset{!}{=} \partial_\mu k^\mu \tag{15}$$

We assume the functions ω_a to have *compact support*, which means that they decrease rapidly at the boundary. This implies that boundary terms containing ω_a vanish, which affects both $\partial_\mu\theta^\mu$ and $\partial_\mu k^\mu$. Inserting the transformation eq. (7) yields,

$$\begin{aligned}
0 = E^k\hat{\delta}\varphi_k &= \mathrm{i}E^k\omega_a(x)F_k^a[\varphi] + \mathrm{i}E^k\partial_\mu\omega_a(x)F_k^{a\mu}[\varphi] && \text{(16a)}\\
&= \mathrm{i}E^k\omega_a(x)F_k^a[\varphi] - \mathrm{i}\omega_a(x)\partial_\mu\Big(E^kF_k^{a\mu}[\varphi]\Big) + \mathrm{i}\partial_\mu\Big(E^k\omega_a(x)F_k^{a\mu}[\varphi]\Big) && \text{(16b)}\\
&\equiv \mathrm{i}E^k\omega_a(x)F_k^a[\varphi] - \mathrm{i}\omega_a(x)\partial_\mu\Big(E^kF_k^{a\mu}[\varphi]\Big) - \partial_\mu S^\mu && \text{(16c)}
\end{aligned}$$

Finally, using the assumption of compact support again to get rid of the term $\partial_\mu S^\mu$ yields $a = 1, .., A$ constraining equations for the equations of motion:

$$E^kF_k^a[\varphi] - \partial_\mu\Big(E^kF_k^{a\mu}[\varphi]\Big) = 0 \tag{17}$$

For a local symmetry, Noether's first theorem still holds: Note that the current S^μ follows the same equation as the full Noether current j^μ. However, S^μ vanishes on-shell, as do charges defined using S^0. Therefore, local symmetries do not yields conserved quantities.

References

[1] E. Noether, *Invariante Variationsprobleme,* Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1918** (1918) 235.

[2] S. G. Avery and B. U. W. Schwab, *Noether's Second Theorem and Ward Identities for Gauge Symmetries,* Journal of High Energy Physics **2016** (2016) 31, arXiv:1510.07038 [hep-th].