PROJEKTARBEIT

Excited Quarkonia in the Extended Linear Sigma Model (eLSM)

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I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have acknowledged all the sources of information which have been used in the thesis.

Contents

Chapter 1

Introduction

Mesons are defined as strongly interacting particles that have integer spin. In terms of the so-called constituent quarks, they can have various structures. For example, many of them are compatible with a $\bar{q}q$ structure. All mesons can be categorized via some of their properties: the spin J and their eigenvalues under parity transformations and charge conjugation, P and C respectively. That way, mesons are described as scalar ($J^{PC} = 0^{++}$), pseudoscalar ($J^{PC} = 0^{-+}$), vector ($J^{PC} = 1^{--}$) and axial-vector ($J^{PC} = 1^{++}$). There are of course mesons with higher spins, but they do not appear in this work.

Mesons emerge from the strong interaction that is described by Quantum Chromodynamics (QCD). QCD is not applicable perturbatively to meson research since its coupling becomes large at small energies where mesons emerge. For this reason, we work with an effective model where mesons are present as degrees of freedom from the beginning. When working with an effective model, one has to identify experimentally measured resonances with certain fields in the model. This is a non trivial task, since there is no obvious one-to-one correspondence. By carefully choosing certain fields to represent physical resonances, it is possible to fit the parameters of the model to reproduce the observed physics and make predictions. The focus of this thesis are excited mesons. The main results will be the prediction of some excited mesons' mass, as well as decay widths of excited mesons decaying into ground state mesons. This plays a part in better understanding the current meson spectrum and connecting it to the current theoretical knowledge of the strong interaction.

Chapter 2 will review quantum chromodynamics, the theory of strong interactions, whose symmetries the model is based on. Chapter 3 will introduce the extended Linear Sigma Model (eLSM), where we include excited mesons. Chapter 4 prepares the theory behind the calculation of decay widths and chapter 5 presents the results of this project work.

Chapter 2

QCD and Its Symmetries

2.1 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is a gauge theory based on the Lie group $SU(3)$, which describes strong interactions. The Lagrangian density in its compact form reads

$$\mathcal{L} = \bar{q}\big[\mathrm{i}\not{D} - \mathsf{M}\big]q - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}, \tag{2.1}$$

or explicitly,

$$\mathcal{L} = \bar{q}^{\alpha}_{f,i}\big[\mathrm{i}\gamma^{\mu}_{\alpha\beta}\partial_\mu\delta_{ff'}\delta_{ij} - g\gamma^{\mu}_{\alpha\beta}\mathsf{t}^a_{ij}A^a_\mu\delta_{ff'} - \mathsf{M}_{ff'}\delta_{ij}\delta_{\alpha\beta}\big]q^{\beta}_{f',j} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}, \tag{2.2}$$

where repeated indices are summed over. The $q^{\alpha}_{f,i}$ are quark spinor fields of flavor f, mass matrix $\mathsf{M} = diag(m_1, \dots, m_{N_f})$ and colour i, with i taking values from 1 to $N_c = 3$. The γ^μ are the Dirac γ-matrices, satisfying the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\}_{\alpha\beta} = 2\eta^{\mu\nu}\delta_{\alpha\beta}, \tag{2.3}$$

with spinor indices α, β, ranging from 1 to 4. The A^a_μ are the $N_c^2 - 1 = 8$ new gauge fields, ensuring local $SU(N_c)$ invariance of eq. (2.2) with a taking values from 1 to $N_c^2 - 1$. The matrices $\mathsf{t}^a_{ij} = \frac{1}{2}\lambda^a_{ij}$ are the generators of $SU(3)$ and λ^a the Gell-Mann-matrices. The parameter g is the coupling between quarks and gluons.
In analogy to QED, the gluon field strength tensors are defined as

$$\begin{aligned} G^a_{\mu\nu}\mathsf{t}^a = \mathbb{G}_{\mu\nu} &= D_\mu\mathcal{A}_\nu - D_\nu\mathcal{A}_\mu \\ &= D_\mu A^a_\nu\mathsf{t}^a - D_\nu A^a_\mu\mathsf{t}^a \\ &= \partial_\mu A^a_\nu\mathsf{t}^a - \partial_\nu A^a_\mu\mathsf{t}^a + \mathrm{i}gA^a_\mu A^b_\nu[\mathsf{t}^a, \mathsf{t}^b] \\ &= \partial_\mu A^a_\nu\mathsf{t}^a - \partial_\nu A^a_\mu\mathsf{t}^a - g\,f^{abc}A^a_\mu A^b_\nu\mathsf{t}^c, \end{aligned} \tag{2.4}$$

where f^{abc} denotes the completely antisymmetric stucture constants of a $SU(3)$ Lie algebra, satisfying

$$[\mathsf{t}^a, \mathsf{t}^b]_{ij} = \mathrm{i}f^{abc}\mathsf{t}^c_{ij}, \tag{2.5}$$

and D_μ is the covariant derivative, that had to be introduced in order to ensure local gauge invariance.

$$\begin{aligned} \partial_\mu\delta_{ij} \to D_{\mu,ij} &= \partial_\mu\delta_{ij} + \mathrm{i}g\mathcal{A}_{\mu,ij} \\ &= \partial_\mu\delta_{ij} + \mathrm{i}gA^a_\mu\mathsf{t}^a_{ij} \end{aligned} \tag{2.6}$$

2.2 Symmetries

Quantum chromodynamics exhibits many symmetries, which will be discussed in the following sections.

2.2.1 Color symmetry

This is the exact $SU(3)_c$ symmetry, the theory was constructed on. It is a local symmetry and gives rise to 8 gluon fields.

2.2.2 Flavor symmetry

A quarks's flavor determines its charge, hypercharge and possible interactions. The different flavors and their masses are arranged in the table below, according to ascending mass.

flavor	mass
up	$2.2^{+0.6}_{-0.4}$ MeV
down	$4.7^{+0.5}_{-0.4}$ MeV
strange	96^{+8}_{-4} MeV
charm	1.27 ± 0.03 GeV
bottom	$4.18^{+0.04}_{-0.03}$ GeV
top	$173.21 \pm 0.51 \pm 0.71$ GeV

TABLE 2.1: Quark flavors and their masses [1].

At the typical low-energy scale of QCD, set by the proton mass $m_N = 939$ MeV, one can consider up and down quarks degenerate. This gives rise to a global $SU(2)$ symmetry. For all other flavors the effects of explicit symmetry breaking are noticeable and it is necessary to consider them in construction of QCD models.

2.2.3 CPT symmetry

Quantum chomodynamics is CPT invariant, as well as C, P and T invariant separately. The next sections cover the discrete symmetries separately [2], using the Dirac representation of spinors and γ matrices.

Parity

A fermionic quantum field, like a quark, transforms under parity as

$$q(x) \xrightarrow{P} U_P^{-1} q(x) U_P \overset{\text{Dirac rep.}}{=} \eta_P \, \gamma_0 \, q(\Lambda_P^{-1} x) \qquad (\Lambda_P^{-1} = \Lambda_P), \tag{2.7}$$

with a unitary transformation $U_P = U(\Lambda_P)$, the Lorentz transformation for a parity transformation,

$$(\Lambda_P)^\mu{}_\nu = diag(1, -1, -1, -1)^\mu{}_\nu, \tag{2.8}$$

and a phase $\eta_P \in \mathbb{C}$ that obeys $\eta_P^2 = \pm 1$, since observable quantities contain an even number of fermion operators and applying P twice should yield 1 as eigenvalue.
A vector field, like the gluon, transforms like

$$\bar{q}(x)\gamma^\mu q(x) \xrightarrow{P} \begin{cases} \bar{q}(\Lambda_P x)\gamma^\mu q(\Lambda_P x) & \mu = 0 \\ -\bar{q}(\Lambda_P x)\gamma^\mu q(\Lambda_P x) & \mu \neq 0 \end{cases} \tag{2.9}$$

Time Reversal

Time reversal acts on a spinor field as an *anti*unitary operator $U_T = U(\Lambda_T)$, $U_T^{-1}\,\mathrm{i}\,U_T = -\mathrm{i}$, where Λ_T, the Lorentz transformation for time reversal, reads

$$\Lambda_T{}^{\mu}{}_{\nu} = diag(-1,1,1,1)^{\mu}{}_{\nu}. \tag{2.10}$$

The quark field transforms like

$$q(x) \xrightarrow{T} U_T^{-1} q(x) U_T \overset{\text{Dirac rep.}}{=} -\eta_T\,\gamma^1\gamma^3\,q(\Lambda_T^{-1}x) \qquad (\Lambda_T^{-1} = \Lambda_T), \tag{2.11}$$

where the phase factor $\eta_T \in \mathbb{C}$ again obeys $\eta_T^2 = \pm 1$. A vector field transforms like

$$\bar{q}(x)\gamma^\mu q(x) \xrightarrow{T} \begin{cases} \bar{q}(\Lambda_T x)\gamma^\mu q(\Lambda_T x) & \mu = 0 \\ -\bar{q}(\Lambda_T x)\gamma^\mu q(\Lambda_T x) & \mu \neq 0 \end{cases}. \tag{2.12}$$

Charge Conjugation

Charge conjugation is usually defined to take a fermion with a certain spin orientation and transform it to an antifermion with the same spin orientation. For a quark spinor field, this yields the transformation

$$q(x) \xrightarrow{C} U_C^{-1} q(x) U_C \overset{\text{Dirac rep.}}{=} -\mathrm{i}\eta_C\,\gamma^2\,q^*(x), \tag{2.13}$$

where $\eta_C \in \mathbb{C}$ is a phase factor with $\eta_C^2 = \pm 1$ and q^* denotes complex conjugation. A vector field transforms like

$$\bar{q}(x)\gamma^\mu q(x) \xrightarrow{C} -\bar{q}(x)\gamma^\mu q(x). \tag{2.14}$$

2.2.4 Chiral symmetry

One defines left- and right-handed projection operators,

$$P_{R,L} = \tfrac{1}{2}(\mathbb{1} \pm \gamma^5), \qquad \gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.15}$$

which decompose a quark into its left-handed and right-handed components:

$$q_f = \mathbb{1} q_f = (P_L + P_R) q_f = q_{f,L} + q_{f,R}, \tag{2.16}$$

where $q_f = \{u, d, c, s, t, b\}$. One can apply this to the quark sector of the Lagrangian eq. (2.2):

$$\mathcal{L}_{\text{quark}} = \bar{q}_{f,L} i\gamma^\mu D_\mu q_{f,L} + \bar{q}_{f,R} i\gamma^\mu D_\mu q_{f,R} - \bar{q}_{f,L} m_f q_{f,R} - \bar{q}_{f,R} m_f q_{f,L}, \tag{2.17}$$

with suppressed color and spinor indices. It follows that in the last two terms, i.e. the mass terms, the left- and right-handed components mix, whereas in the first two terms, i.e. kinetic terms, this is not the case.

One now defines transformations that only concern a certain chirality of quarks:

$$q_{f,L} \to q'_{f,L} = U_L q_{f,L} \quad \text{and} \quad q_{f,R} \to q'_{f,R} = U_R q_{f,R}, \tag{2.18}$$

with a unitary transformation

$$U_L = \exp(-\mathrm{i}\alpha_L^a \mathrm{t}^a) \quad \text{and} \quad U_R = \exp(-\mathrm{i}\alpha_R^a \mathrm{t}^a), \tag{2.19}$$

where $\alpha_{L,R}$ is a real parameter.
Omitting the mass term in eq. (2.17), this is a global $SU(N_f)_L \otimes SU(N_f)_R$ symmetry, called the chiral symmetry.

By defining *vector* and *axial-vector* transformations via

$$U_V = U_L = U_R \quad \text{and} \quad U_A = U_L = U_R^\dagger, \tag{2.20}$$

one can write

$$U(N_f)_L \otimes U(N_f)_R \equiv U(N_f)_V \otimes U(N_f)_A \tag{2.21}$$
$$\equiv U(1)_V \otimes SU(N_f)_V \otimes U(1)_A \otimes SU(N_f)_A. \tag{2.22}$$

These symmetries hold under certain conditions:

- $U(1)_V$: exact, gives rise to a conserved Noether charge, called the baryon number
- $SU(N_f)_V$: explicitly broken by non-degenerate quark masses
- $U(1)_A$: explicitly broken for non-vanishing quark masses and exhibits an anomaly
- $SU(N_f)_A$: explicitly broken for non-vanishing quark masses; it is also spontaneously broken, giving rise to $N_f^2 - 1$ Goldstone bosons

2.2.5 Further symmetries

For completeness, QCD also exhibits the following symmetries: Z-symmetry (spontaneously broken in the gauge sector of QCD at finite temperatures; many confinement studies are based on this theory) and conformal symmetry (broken by the so called „trace anomaly“).

Chapter 3

The Linear Sigma Model

3.1 Degrees of Freedom

It feels natural to consider quarks, antiquarks and gluons to be the degrees of freedom to work with. This happens in Lattice QCD, where one simulates a discretized space-time lattice. On this lattice, quarks and antiquarks live on fixed spacetime points and gluons propagate between them. One then tries to let the lattice spacing go to zero, in order to simulate a continuum. This is one example of "QCD from first principles".

Another possibility is to look at the low energy spectrum, where mesons and baryons live, and consider them to be the degrees of freedom. By doing so, one is using an "effective model".

3.2 Constructing the Model

The model used in this work is the "Linear Sigma Model" [3]. It is based on the chiral symmetry and the starting point is to define a meson matrix Φ_{ij},

$$\Phi_{ij} = \sqrt{2}\bar{q}_{f_j,R}q_{f_i,L}. \tag{3.1}$$

After some Clifford algebra, this matrix can be split up into two parts,

$$\begin{aligned}
\Phi_{ij} &= \sqrt{2}\bar{q}_{f_j,R}q_{f_i,L} && \text{(3.2a)}\\
&= \sqrt{2}q^\dagger_{f_j,R}\gamma^0 q_{f_i,L} && \text{(3.2b)}\\
&= \sqrt{2}\big(P_R q_{f_j}\big)^\dagger \gamma^0 P_L q_{f_i} && \text{(3.2c)}\\
&= \sqrt{2}q^\dagger_{f_j}\gamma^0 P_L^2 q_{f_i} && \text{(3.2d)}\\
&= \sqrt{2}\bar{q}_{f_j}P_L q_{f_i} && \text{(3.2e)}\\
&= \tfrac{1}{\sqrt{2}}\bar{q}_{f_j}\big(\mathbb{1}-\gamma^5\big)q_{f_i} && \text{(3.2f)}\\
&= \tfrac{1}{\sqrt{2}}\Big[\bar{q}_{f_j}q_{f_i} + \mathrm{i}\bar{q}_{f_j}\gamma^5\mathrm{i}q_{f_i}\Big] && \text{(3.2g)}\\
&\equiv \tfrac{1}{\sqrt{2}}\big(S+\mathrm{i}P\big). && \text{(3.2h)}
\end{aligned}$$

Φ_{ij} is now represented by a scalar meson matrix S and a pseudoscalar meson matrix P.

For $N_f = 2$, S and P read

$$S \equiv \begin{pmatrix} \bar{u}u & \bar{d}u \\ \bar{u}d & \bar{d}d \end{pmatrix}, \qquad P \equiv \begin{pmatrix} \bar{u}\gamma^5 \mathrm{i}u & \bar{d}\gamma^5 \mathrm{i}u \\ \bar{u}\gamma^5 \mathrm{i}d & \bar{d}\gamma^5 \mathrm{i}d \end{pmatrix}. \tag{3.3}$$

The important part here is to identify the states in S and P with mesons. The lightest scalar mesons are the σ (appears as isosinglet) and the $\mathbf{a}_0$ (this is an isotriplet, meaning there are a_0^+, a_0^-, a_0^0). Comparing this to the structures we can read off S, one identifies[1]

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(\sigma_N + a_0^0) & a_0^+ \\ a_0^- & \frac{1}{\sqrt{2}}(\sigma_N - a_0^0) \end{pmatrix}. \tag{3.4}$$

Next, one identifies pseudo scalar mesons, yielding

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(\eta_N + \pi^0) & \pi^+ \\ \pi^- & \frac{1}{\sqrt{2}}(\eta_N - \pi^0) \end{pmatrix}. \tag{3.5}$$

Similarly, (axial-)vector mesons appearing in the vector meson matrix V^μ and the axial-vector meson matrix A^μ are introduced,

$$V^\mu = \bar{q}\gamma^\mu q, \quad A^\mu = \bar{q}\gamma^\mu\gamma_5 q, \tag{3.6}$$

and combined into

$$L^\mu = V^\mu + A^\mu, \tag{3.7}$$
$$R^\mu = V^\mu - A^\mu, \tag{3.8}$$

in order for them to have a well defined chiral transformation (see below).

3.2.1 Building a Lagrangian

From the definition of the meson matrix Φ in (3.1), one can calculate its transformation under the chiral symmetry:

$$\begin{aligned}
\Phi_{ij} &= \sqrt{2}\bar{q}_{f_j,R} q_{f_i,L} && (3.9\text{a})\\
&= \sqrt{2} q^\dagger_{f_j,R}\gamma^0 q_{f_i,L} && (3.9\text{b})\\
&\to \sqrt{2}\big(U_R q_{f_j,R}\big)^\dagger \gamma^0 U_L q_{f_i,L} && (3.9\text{c})\\
&= \sqrt{2} q^\dagger_{f_j,R} U_R^\dagger \gamma^0 U_L q_{f_i,L} && (3.9\text{d})\\
&= U_L \sqrt{2}\bar{q}_{f_j,R} q_{f_i,L} U_R^\dagger && (3.9\text{e})\\
&= U_L \Phi_{ij} U_R^\dagger. && (3.9\text{f})
\end{aligned}$$

Similarly, the vector mesons transform like

$$L_\mu \to U_L L_\mu U_L^\dagger, \tag{3.10}$$
$$R_\mu \to U_R R_\mu U_R^\dagger. \tag{3.11}$$

One can easily check that expressions like

$$\mathrm{Tr}\Big(\Phi^\dagger\Phi\Big), \quad \mathrm{Tr}\Big(\Phi^\dagger\Phi\Phi^\dagger\Phi\Big), \quad \mathrm{Tr}\Big(L_\mu \Phi R^\mu \Phi^\dagger\Big), \quad \text{etc.} \tag{3.12}$$

[1] Since this is an example with two flavors only, the strange part of the sigma and eta meson gets omitted, therefore denoting them as their non-strange part σ_N and η_N.

are invariant under transformations (3.9c), because of the possibility of cyclic permutations in a trace. These are the building blocks of the Lagrangian. Every term is invariant under C, P and T transformations separately, like the QCD Lagrangian, eq. (2.1), is. As an example, the meson matrix Φ_{ij} transforms under parity, eq. (2.7), as

$$\Phi_{ij} = \sqrt{2}\bar{q}_{j,R}(x) q_{i,L}(x) \tag{3.13a}$$
$$= \sqrt{2}\big[P_R q_j(x)\big]^\dagger \gamma_0 P_L q_i(x) \tag{3.13b}$$
$$\xrightarrow{P} \sqrt{2}\Big[P_R \eta_P \gamma_0 q_j(\Lambda_P x)\Big]^\dagger \gamma_0 P_L \eta_P \gamma_0 q_i(\Lambda_P x) \tag{3.13c}$$
$$= \sqrt{2}|\eta_P|^2 q_j^\dagger(\Lambda_P x)\gamma_0 P_R^\dagger \gamma_0 P_L \gamma_0 q_i(\Lambda_P x) \tag{3.13d}$$
$$= \sqrt{2} q_{j,L}^\dagger(\Lambda_P x)\gamma_0 q_{i,R}(\Lambda_P x) \tag{3.13e}$$
$$= \Big[\sqrt{2} q_{i,L}^\dagger(\Lambda_P x)\gamma_0^\dagger q_{j,R}(\Lambda_p x)\Big]^\dagger \tag{3.13f}$$
$$= \Phi_{ij}^\dagger. \tag{3.13g}$$

To get eq. (3.13f), we used $P_{R,L}\gamma_0 = \gamma_0 P_{L,R}$. Therefore, a term like the first one in eq. (3.12) is invariant under a parity transformation P:

$$\mathrm{Tr}\Big[\Phi^\dagger\Phi\Big] = \Phi_{ij}^\dagger \Phi_{ji} \xrightarrow{P} \Phi_{ij}\Phi_{ji}^\dagger = \mathrm{Tr}\Big[\Phi\Phi^\dagger\Big] = \mathrm{Tr}\Big[\Phi^\dagger\Phi\Big]. \tag{3.14}$$

3.3 Extending the Model

A generalization of this model is called the "extended Linear Sigma Model (eLSM)". In this work, the model gets extended by excited scalar and pseudoscalar mesons. One introduces the excited meson matrix $\Phi_{E\,ij}$, which is built in analogy to Φ_{ij}, but without a vacuum expectation value for the $J^{PC} = 0^{++}$ meson. The behaviour under a chiral transformation, or C, P, T transformations is as well in analogy to the ground state meson matrix:

$$\Phi_E \xrightarrow{\text{chiral}} U_L \Phi_E U_R^\dagger. \tag{3.15}$$

The introduction of a new matrix in this model leads to several new term in the Lagrangian, such as

$$\mathrm{Tr}\Big[\Phi_E^\dagger \Phi + \Phi^\dagger \Phi_E\Big], \quad \mathrm{Tr}\Big[L_\mu \Phi_E R^\mu \Phi_E^\dagger\Big]. \tag{3.16}$$

3.3.1 The Lagrangian

This is the full Lagrangian used in this thesis:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_E, \tag{3.17}$$

with the ground-state Lagrangian,

$$\mathcal{L}_0 = \mathrm{Tr}\Big[(D^\mu\Phi)^\dagger(D_\mu\Phi)\Big] - m_0^2\,\mathrm{Tr}\Big[\Phi^\dagger\Phi\Big] - \lambda_1\big(\mathrm{Tr}\Big[\Phi^\dagger\Phi\Big]\big)^2 - \lambda_2\,\mathrm{Tr}\Big[(\Phi^\dagger\Phi)^2\Big]$$
$$-\frac{1}{4}\mathrm{Tr}\big[L_{\mu\nu}^2 + R_{\mu\nu}^2\big] + \mathrm{Tr}\Big[\Big(\frac{m_1^2}{2} + \Delta\Big)(L_\mu^2 + R_\mu^2)\Big] + \mathrm{Tr}\Big[H(\Phi + \Phi^\dagger)\Big]$$

$$
\begin{aligned}
&+ \mathrm{Tr}\Big[\Phi^\dagger\Phi\delta + \Phi\Phi^\dagger\delta\Big] + c_1(\det\Phi - \det\Phi^\dagger)^2 \\
&+ \mathrm{i}\frac{g_2}{2}\big(\mathrm{Tr}[L_{\mu\nu}[L^\mu, L^\nu]] + \mathrm{Tr}[R_{\mu\nu}[R^\mu, R^\nu]]\big) + \frac{h_1}{2}\mathrm{Tr}\Big[\Phi^\dagger\Phi\Big]\mathrm{Tr}\big[L_\mu^2 + R_\mu^2\big] \\
&+ h_2\,\mathrm{Tr}\big[|L_\mu\Phi|^2 + |\Phi R_\mu|^2\big] + 2h_3\,\mathrm{Tr}\Big[L_\mu\Phi R^\mu\Phi^\dagger\Big] \\
&+ g_3\big(\mathrm{Tr}[L_\mu L_\nu L^\mu L^\nu] + \mathrm{Tr}[R_\mu R_\nu R^\mu R^\nu]\big) \\
&+ g_4\big(\mathrm{Tr}[L_\mu L^\mu L_\nu L^\nu] + \mathrm{Tr}[R_\mu R^\mu R_\nu R^\nu]\big) + g_5\,\mathrm{Tr}[L_\mu L^\mu]\,\mathrm{Tr}[R_\mu R^\mu] \\
&+ g_6\big(\mathrm{Tr}[L_\mu L^\mu]\,\mathrm{Tr}[L_\nu L^\nu] + \mathrm{Tr}[R_\mu R^\mu]\,\mathrm{Tr}[R_\nu R^\nu]\big),
\end{aligned}
\tag{3.18}
$$

and the Lagrangian with excited terms,

$$
\begin{aligned}
\mathcal{L}_E = {}& \mathrm{Tr}\Big[(D_\mu\Phi_E)^\dagger(D^\mu\Phi_E)\Big] + \alpha\,\mathrm{Tr}\Big[(D_\mu\Phi)^\dagger(D^\mu\Phi_E) + (D_\mu\Phi_E)^\dagger(D^\mu\Phi)\Big] \\
&- (m_0^\star)^2\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\Big] - \lambda_0\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi + \Phi^\dagger\Phi_E\Big] - \lambda_1^\star\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\Big]\mathrm{Tr}\Big[\Phi^\dagger\Phi\Big] \\
&- \lambda_2^\star\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\Phi^\dagger\Phi + \Phi_E\Phi_E^\dagger\Phi\Phi^\dagger\Big] - \kappa_1\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi + \Phi^\dagger\Phi_E\Big]\mathrm{Tr}\Big[\Phi^\dagger\Phi\Big] \\
&- \kappa_2\Big(\mathrm{Tr}\Big[\Phi_E^\dagger\Phi + \Phi^\dagger\Phi_E\Big]\Big)^2 - \kappa_3\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi + \Phi^\dagger\Phi_E\Big]\mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\Big] \\
&- \kappa_4\Big(\mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\Big]\Big)^2 - \xi_1\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi\Phi^\dagger\Phi + \Phi_E\Phi^\dagger\Phi\Phi^\dagger\Big] \\
&- \xi_2\,\mathrm{Tr}\Big[\Phi_E^\dagger\Phi\Phi_E^\dagger\Phi + \Phi^\dagger\Phi_E\Phi^\dagger\Phi_E\Big] - \xi_3\,\mathrm{Tr}\Big[\Phi^\dagger\Phi_E\Phi_E^\dagger\Phi_E + \Phi\Phi_E^\dagger\Phi_E\Phi_E^\dagger\Big] \\
&- \xi_4\,\mathrm{Tr}\Big[(\Phi_E^\dagger\Phi_E)^2\Big] + \mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\delta_E + \Phi_E\Phi_E^\dagger\delta_E\Big] \\
&+ c_1^\star\big[(\det\Phi - \det\Phi_E^\dagger)^2 + (\det\Phi^\dagger - \det\Phi_E)^2\big] + c_{1E}^\star(\det\Phi_E - \det\Phi_E^\dagger)^2 \\
&+ \frac{h_1^\star}{2}\mathrm{Tr}\Big[\Phi_E^\dagger\Phi + \Phi^\dagger\Phi_E\Big]\mathrm{Tr}\big[L_\mu^2 + R_\mu^2\big] + \frac{h_{1E}^\star}{2}\mathrm{Tr}\Big[\Phi_E^\dagger\Phi_E\Big]\mathrm{Tr}\big[L_\mu^2 + R_\mu^2\big] \\
&+ h_2^\star\,\mathrm{Tr}\Big[\Phi_E^\dagger L_\mu L^\mu\Phi + \Phi^\dagger L_\mu L^\mu\Phi_E + R_\mu\Phi_E^\dagger\Phi R^\mu + R_\mu\Phi^\dagger\Phi_E R^\mu\Big] \\
&+ h_{2E}^\star\,\mathrm{Tr}\big[|L_\mu\Phi_E|^2 + |\Phi_E R_\mu|^2\big] + 2h_3^\star\,\mathrm{Tr}\Big[L_\mu\Phi_E R^\mu\Phi^\dagger + L_\mu\Phi R^\mu\Phi_E^\dagger\Big] \\
&+ 2h_{3E}^\star\,\mathrm{Tr}\Big[L_\mu\Phi_E R^\mu\Phi_E^\dagger\Big].
\end{aligned}
\tag{3.19}
$$

Here,

$$
\begin{aligned}
D_\mu\Phi &= \partial_\mu\Phi - \mathrm{i}g_1(L_\mu\Phi - \Phi R_\mu), \\
D_\mu\Phi_E &= \partial_\mu\Phi_E - \mathrm{i}g_1^\star(L_\mu\Phi_E - \Phi_E R_\mu), \\
L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu, \\
R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu.
\end{aligned}
$$

For $N_f = 3$, the scalar meson matrix reads

$$
\Phi = \sum_{i=0}^{8}(S_i + \mathrm{i}P_i)T_i = \frac{1}{\sqrt{2}}\begin{pmatrix}
\frac{(\sigma_N + a_0^0) + \mathrm{i}(\eta_N + \pi^0)}{\sqrt{2}} & a_0^+ + \mathrm{i}\pi^+ & K_S^+ + \mathrm{i}K^+ \\
a_0^- + \mathrm{i}\pi^- & \frac{(\sigma_N - a_0^0) + \mathrm{i}(\eta_N - \pi^0)}{\sqrt{2}} & K_S^0 + \mathrm{i}K^0 \\
K_S^- + \mathrm{i}K^- & \bar{K}_S^0 + \mathrm{i}\bar{K}^0 & \sigma_S + \mathrm{i}\eta_S
\end{pmatrix},
$$

and the vector matrices read

$$
L^\mu = \sum_{i=0}^{8}(V_i^\mu + A_i^\mu)T_i = \frac{1}{\sqrt{2}}\begin{pmatrix}
\frac{\omega_N + \rho^0}{\sqrt{2}} + \frac{f_{1N} + a_1^0}{\sqrt{2}} & \rho^+ + a_1^+ & K^{\star +} + K_1^+ \\
\rho^- + a_1^- & \frac{\omega_N - \rho^0}{\sqrt{2}} + \frac{f_{1N} - a_1^0}{\sqrt{2}} & K^{\star 0} + K_1^0 \\
K^{\star -} + K_1^- & \bar{K}^{\star 0} + \bar{K}_1^0 & \omega_S + f_{1S}
\end{pmatrix}^\mu,
$$

$$R^\mu = \sum_{i=0}^{8}(V_i^\mu - A_i^\mu)T_i = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\omega_N+\rho^0}{\sqrt{2}} - \frac{f_{1N}+a_1^0}{\sqrt{2}} & \rho^+ - a_1^+ & K^{\star+} - K_1^+ \\ \rho^- - a_1^- & \frac{\omega_N-\rho^0}{\sqrt{2}} - \frac{f_{1N}-a_1^0}{\sqrt{2}} & K^{\star 0} - K_1^0 \\ K^{\star-} - K_1^- & \bar{K}^{\star 0} - \bar{K}_1^0 & \omega_S - f_{1S} \end{pmatrix}^\mu ,$$

with T_i the generators of $U(3)$. Next to the usual kinetic and mass terms in a Lagrangian, there are several more terms in eqs. (3.18) and (3.19) that play a certain role:

Description	Parameter
self-interaction of ground state (pseudo)scalars	$\lambda_{1,2}$
explicit symmetry breaking	H, δ, δ_E
chiral anomaly	c_1, $c_1^\star$, $c_{1E}^\star$
self-interaction of (axial-)vectors or interaction of (axial-)vectors with (pseudo)scalars	$g_{2,3,4,5,6}$
interaction of ground-state (pseudo)scalars with (axial-)vectors	$h_{1,2,3}$
mixing term of ground-state and excited (pseudo)scalars	α
interaction of ground-state and excited (pseudo)scalars	λ_0, $\lambda_{1,2}^\star$, $\kappa_{1,2,3,4}$, $\xi_{1,2,3,4}$
interaction of (pseudo)scalars (gs. and ex.) with (axial-)vectors	$h^\star_{1,1E,2,2E,3,3E}$

Table 3.1: Summary of coefficients in the Lagrangian.

The matrices corresponding to explicit symmetry breaking read $H = diag(h_N, h_N, h_S)$, $\Delta = diag(0,0,\Delta_S)$, $\delta = diag(0,0,\delta_S)$, $\delta_E = diag(0,0,\delta_{E,S})$ and describe non-vanishing quark masses. In this model, we include spontaneous chiral symmetry breaking by introducing the vacuum expectation value of the two scalar, isospin-0 states σ_N and σ_S. Shifting these two mesons $\sigma_{N,S} \to \sigma_{N,S} + \phi_{N,S}$ leads to mixing terms of spin-0 (S) and spin-1 (V) fields. They can be removed by shifting the spin-1 fields by

$$V^\mu \to V^\mu + Z_S w_V \partial^\mu S. \tag{3.20}$$

The explicit values of the new introduced constants can be found in [3].

Chapter 4

Calculating Decay Widths

4.1 Theoretical Background

The decay width for a n-body decay is given by [1]

$$\mathrm{d}\Gamma = \frac{(2\pi)^4}{2M}|-\mathrm{i}\mathcal{M}|^2\mathrm{d}\Phi_n, \tag{4.1}$$

where M is the decaying particle's mass, $-\mathrm{i}\mathcal{M}$ is the decay amplitude and $\mathrm{d}\Phi_n$ is an infinitesimal element of a n-body phase space,

$$\mathrm{d}\Phi_n = \delta^{(4)}(P - \sum_{i=1}^{n} K_i)\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3 2E_i}, \tag{4.2}$$

where P is the four-momentum of the decaying particle, $K_i = (E_i, \mathbf{k}_i)$ are the four-momenta of the resulting particles and $\delta^{(4)}(X)$ is the four-dimensional δ function.

This work will only consider decay amplitudes $|-\mathrm{i}\mathcal{M}|$ that do not depend on the angles of the scattering process.

4.1.1 Two body decay

For a two body decay, the phase space element $\mathrm{d}\Phi_2$ reads

$$\mathrm{d}\Phi_2 = \delta^{(4)}(P - K_1 - K_2)\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3 2E^{(1)}_{\mathbf{k}_1}}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3 2E^{(2)}_{\mathbf{k}_2}}. \tag{4.3}$$

In the center-of-mass system, the momenta read

$$P^\mu = \begin{pmatrix} M \\ \mathbf{0} \end{pmatrix}^\mu, \quad K_1^\mu = \begin{pmatrix} E^{(1)}_{\mathbf{k}_1} \\ \mathbf{k}_1 \end{pmatrix}^\mu, \quad K_2^\mu = \begin{pmatrix} E^{(2)}_{\mathbf{k}_2} \\ \mathbf{k}_2 \end{pmatrix}^\mu, \tag{4.4}$$

with $E^{(i)}_{\mathbf{k}_j} = \sqrt{m_i^2 + \mathbf{k}_j^2}$ and thus the four dimensional δ function can be split up into

$$\delta^{(4)}(P - K_1 - K_2) = \delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2})\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2). \tag{4.5}$$

By using a well-known identity for the Dirac δ function,

$$\delta\big(g(x)\big) = \sum \frac{\delta(x - x_i)}{|g'(x_i)|}, \quad \text{with } g(x_i) = 0 \quad \forall i, \tag{4.6}$$

the δ function in eq. (4.5) becomes

$$\delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2}) = \frac{1}{k_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \delta(|\mathbf{k}_1| - k_f), \tag{4.7}$$

where k_f is a consequence of energy-momentum conservation:

$$k_f \equiv \frac{1}{2M} \sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}. \tag{4.8}$$

The decay with, eq. (4.1), now yields

$$\begin{aligned} d\Gamma &= \frac{(2\pi)^4}{2M} |-\mathrm{i}\mathcal{M}|^2 \frac{1}{k_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \delta(|\mathbf{k}_1| - k_f) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3 2E^{(1)}_{\mathbf{k}_1}} \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3 2E^{(2)}_{\mathbf{k}_2}} \\ &= \frac{|-\mathrm{i}\mathcal{M}|^2}{(2\pi)^2 8Mk_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \frac{\mathrm{d}^3\mathbf{k}_1}{E^{(1)}_{\mathbf{k}_1}} \frac{\mathrm{d}^3\mathbf{k}_2}{E^{(2)}_{\mathbf{k}_2}} \delta(|\mathbf{k}_1| - k_f)\, \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2). \end{aligned} \tag{4.9}$$

The integration over $\mathbf{k}_2$ can be easily done using the three dimensional δ function,

$$\begin{aligned} \Gamma &= \frac{|-\mathrm{i}\mathcal{M}|^2}{(2\pi)^2 8Mk_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \int \frac{\mathrm{d}^3\mathbf{k}_1}{E^{(1)}_{\mathbf{k}_1}} \frac{\mathrm{d}^3\mathbf{k}_2}{E^{(2)}_{\mathbf{k}_2}} \delta(|\mathbf{k}_1| - k_f)\, \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \\ &= \frac{|-\mathrm{i}\mathcal{M}|^2}{(2\pi)^2 8Mk_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \int \frac{\mathrm{d}^3\mathbf{k}_1}{E^{(1)}_{\mathbf{k}_1} E^{(2)}_{\mathbf{k}_1}} \delta(|\mathbf{k}_1| - k_f), \end{aligned} \tag{4.10}$$

and the second integration is performed using spherical coordinates, $\mathrm{d}^3\mathbf{k} = |\mathbf{k}|^2 \mathrm{d}|\mathbf{k}| \mathrm{d}\Omega$,

$$\Gamma = \frac{|-\mathrm{i}\mathcal{M}|^2}{(2\pi)^2 8Mk_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \int \frac{|\mathbf{k}_1|^2 \mathrm{d}|\mathbf{k}_1| \mathrm{d}\Omega}{E^{(1)}_{\mathbf{k}_1} E^{(2)}_{\mathbf{k}_1}} \delta(|\mathbf{k}_1| - k_f) \tag{4.11}$$

$$= \frac{|-\mathrm{i}\mathcal{M}|^2}{(2\pi)^2 8Mk_f} \frac{E^{(1)}_{k_f} E^{(2)}_{k_f}}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} \frac{k_f^2}{E^{(1)}_{k_f} E^{(2)}_{k_f}} \int \mathrm{d}\Omega. \tag{4.12}$$

Three-momentum conservation in eq. (4.4) implies that the resulting particles propagate in opposite directions. By choosing a coordinate system, which aligns to one of the particles' path, the $\mathrm{d}\Omega$ integration becomes trivial, $\int \mathrm{d}\Omega = 4\pi$, and the decay width reads

$$\Gamma = \frac{k_f}{8\pi M} \frac{1}{E^{(1)}_{k_f} + E^{(2)}_{k_f}} |-\mathrm{i}\mathcal{M}|^2. \tag{4.13}$$

Energy conservation in eq. (4.4) reduces $E^{(1)}_{k_f} + E^{(2)}_{k_f}$ to the total energy of the decaying particle, which, in its rest frame, is its mass M,

$$\Gamma_{2\ \text{body}} = \frac{k_f}{8\pi M^2} |-\mathrm{i}\mathcal{M}|^2. \tag{4.14}$$

4.1.2 Three body decay

For a three body decay, the phase space element $\mathrm{d}\Phi_3$ reads

$$\mathrm{d}\Phi_3 = \delta^{(4)}(P - K_1 - K_2 - K_3)\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3 2E^{(1)}_{\mathbf{k}_1}}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3 2E^{(2)}_{\mathbf{k}_2}}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3 2E^{(3)}_{\mathbf{k}_3}}. \tag{4.15}$$

The particles' momenta in the center-of-mass system are denoted by

$$P^\mu = \begin{pmatrix} M \\ \mathbf{0} \end{pmatrix}^\mu, \quad K_1^\mu = \begin{pmatrix} E^{(1)}_{\mathbf{k}_1} \\ \mathbf{k}_1 \end{pmatrix}^\mu, \quad K_2^\mu = \begin{pmatrix} E^{(2)}_{\mathbf{k}_2} \\ \mathbf{k}_2 \end{pmatrix}^\mu, \quad K_3^\mu = \begin{pmatrix} E^{(3)}_{\mathbf{k}_3} \\ \mathbf{k}_3 \end{pmatrix}^\mu, \tag{4.16}$$

with $E^{(i)}_{\mathbf{k}_j} = \sqrt{m_i^2 + \mathbf{k}_j^2}$ so that the four dimensional δ function can be split up into

$$\delta^{(4)}(P - K_1 - K_2 - K_3) = \delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2} - E^{(3)}_{\mathbf{k}_3})\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \tag{4.17}$$

The three-dimensional δ function can be used to eliminate one $\mathrm{d}^3\mathbf{k}$ integral, yielding

$$\begin{aligned}\Gamma &= \int \mathrm{d}\Gamma \\ &= \frac{1}{(2\pi)^5 16M}\int \frac{\mathrm{d}^3\mathbf{k}_1\mathrm{d}^3\mathbf{k}_2\mathrm{d}^3\mathbf{k}_3|-\mathrm{i}\mathcal{M}|^2}{E^{(1)}_{\mathbf{k}_1}E^{(2)}_{\mathbf{k}_2}E^{(3)}_{\mathbf{k}_3}}\delta^{(4)}(P - K_1 - K_2 - K_3) \\ &= \frac{1}{(2\pi)^5 16M}\int \frac{\mathrm{d}^3\mathbf{k}_1\mathrm{d}^3\mathbf{k}_2|-\mathrm{i}\mathcal{M}|^2}{E^{(1)}_{\mathbf{k}_1}E^{(2)}_{\mathbf{k}_2}E^{(3)}_{\mathbf{k}_1+\mathbf{k}_2}}\delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2} - E^{(3)}_{\mathbf{k}_1+\mathbf{k}_2}).\end{aligned} \tag{4.18}$$

Now changing to spherical coordinates,

$$\mathrm{d}^3\mathbf{k}_i = |\mathbf{k}_i|^2\mathrm{d}|\mathbf{k}_i|\sin\vartheta_i\mathrm{d}\vartheta_i\mathrm{d}\varphi_i, \tag{4.19}$$

and expressing ϑ_2 as the relative angle to ϑ_1,

$$\sin\vartheta_2\mathrm{d}\vartheta_2 \to \sin\vartheta_{12}\mathrm{d}\vartheta_{12}, \tag{4.20}$$

leaves a trivial integration $\int \sin\vartheta_1\mathrm{d}\vartheta_1\mathrm{d}\varphi_1\mathrm{d}\varphi_2 = 8\pi^2$,

$$\begin{aligned}\Gamma = \frac{1}{(2\pi)^3 8M}\int &\frac{|\mathbf{k}_1|^2|\mathbf{k}_2|^2\mathrm{d}|\mathbf{k}_1|\mathrm{d}|\mathbf{k}_2|\sin\vartheta_{12}\mathrm{d}\vartheta_{12}}{E^{(1)}_{\mathbf{k}_1}E^{(2)}_{\mathbf{k}_2}E^{(3)}_{\mathbf{k}_1+\mathbf{k}_2}} \\ &\times|-\mathrm{i}\mathcal{M}|^2\delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2} - E^{(3)}_{\mathbf{k}_1+\mathbf{k}_2}).\end{aligned} \tag{4.21}$$

Using a small trick, the $\mathrm{d}\vartheta_{12}$ integration will be replaced by a $\mathrm{d}\mathbf{k}_3$ integration: For a given $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$, $\mathbf{k}_3$ will depend only on ϑ_{12},

$$\mathbf{k}_3^2 = (\mathbf{k}_1 + \mathbf{k}_2)^2 = \mathbf{k}_1^2 + \mathbf{k}_2^2 + 2|\mathbf{k}_1||\mathbf{k}_2|\cos\vartheta_{12} \tag{4.22a}$$

$$\begin{aligned}\to \quad 2|\mathbf{k}_3|\mathrm{d}|\mathbf{k}_3| &= 2|\mathbf{k}_1||\mathbf{k}_2|\,\mathrm{d}(\cos\vartheta_{12}) \\ &= -2|\mathbf{k}_1||\mathbf{k}_2|\sin\vartheta_{12}\mathrm{d}\vartheta_{12}.\end{aligned} \tag{4.22b}$$

The decay width now reads[1]

$$\Gamma = \frac{1}{(2\pi)^3 8M}\int \frac{|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}_3|\mathrm{d}|\mathbf{k}_1|\mathrm{d}|\mathbf{k}_2|\mathrm{d}|\mathbf{k}_3|}{E^{(1)}_{\mathbf{k}_1}E^{(2)}_{\mathbf{k}_2}E^{(3)}_{\mathbf{k}_3}}$$

[1] The negative sign gets absorbed into the boundaries of the ϑ integration since $\int_{-1}^{1}\mathrm{d}(\cos\vartheta) = -\int_{\pi}^{0}\sin\vartheta\mathrm{d}\vartheta = \int_0^\pi \sin\vartheta\mathrm{d}\vartheta$.

$$\times |-\mathrm{i}\mathcal{M}|^2\delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2} - E^{(3)}_{\mathbf{k}_3}). \tag{4.23}$$

The next substitution changes the $\mathrm{d}|\mathbf{k}_i|$ integration into a $\mathrm{d}E_i$ integration via

$$E^{(i)}_{\mathbf{k}_i} = \sqrt{m_i^2 + \mathbf{k}_i^2} \tag{4.24}$$

$$\mathrm{d}E^{(i)}_{\mathbf{k}_i} = \frac{1}{2}\frac{1}{\sqrt{m_i^2 + \mathbf{k}_i^2}} 2|\mathbf{k}_i|\mathrm{d}|\mathbf{k}_i| \quad \rightarrow \quad E^{(i)}_{\mathbf{k}_i}\mathrm{d}E^{(i)}_{\mathbf{k}_i} = |\mathbf{k}_i|\mathrm{d}|\mathbf{k}_i|, \tag{4.25}$$

with $i = \{1, 2, 3\}$. Applying this to the decay width in eq. (4.23) yields

$$\Gamma = \frac{1}{(2\pi)^3 8M}\int \mathrm{d}E^{(1)}_{\mathbf{k}_1}\mathrm{d}E^{(2)}_{\mathbf{k}_2}\mathrm{d}E^{(3)}_{\mathbf{k}_3}|-\mathrm{i}\mathcal{M}|^2\delta(M - E^{(1)}_{\mathbf{k}_1} - E^{(2)}_{\mathbf{k}_2} - E^{(3)}_{\mathbf{k}_3}). \tag{4.26}$$

Again, one integration can be evaluated using the δ function,

$$\Gamma = \frac{1}{(2\pi)^3 8M}\int |-\mathrm{i}\mathcal{M}|^2 \mathrm{d}E^{(1)}_{\mathbf{k}_1}\mathrm{d}E^{(3)}_{\mathbf{k}_3}. \tag{4.27}$$

One last substitution changes the $\mathrm{d}E$ integration to a new variable $\mathrm{d}\left(m_{ij}^2\right)$, defined via

$$\begin{aligned} m_{12}^2 &\equiv (K_1^\mu + K_2^\mu)^2 & (4.28)\\ &= (E^{(1)}_{\mathbf{k}_1} + E^{(2)}_{\mathbf{k}_2})^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2 \\ &= (M - E^{(3)}_{\mathbf{k}_3})^2 - \mathbf{k}_3^2 \\ &= M^2 - 2ME^{(3)}_{\mathbf{k}_3} + (E^{(3)}_{\mathbf{k}_3})^2 - \mathbf{k}_3^2 \\ &= M^2 + m_3^2 - 2ME^{(3)}_{\mathbf{k}_3} \\ \rightarrow \quad \mathrm{d}\left(m_{12}^2\right) &= -2M\mathrm{d}E^{(3)}_{\mathbf{k}_3}, & (4.29) \end{aligned}$$

and similarly,

$$\mathrm{d}\left(m_{23}^2\right) = -2M\mathrm{d}E^{(1)}_{\mathbf{k}_1}. \tag{4.30}$$

Using the new variables in eq. (4.27) yields

$$\Gamma_{3\ \mathrm{body}} = \frac{1}{(2\pi)^3 32M^3}\int |-\mathrm{i}\mathcal{M}|^2\,\mathrm{d}\left(m_{12}^2\right)\mathrm{d}\left(m_{23}^2\right). \tag{4.31}$$

The last step is to find the boundaries for the remaining two integrations. m_{12}^2 takes on its smallest value for $\mathbf{k}_1 = \mathbf{k}_2 = 0$ and its largest value for $\mathbf{k}_3 = 0$:

$$(m_{12}^2)_{\mathrm{min}} = (m_1 + m_2)^2, \tag{4.32a}$$

$$(m_{12}^2)_{\mathrm{max}} = (M - m_3)^2. \tag{4.32b}$$

Then, for a given value of m_{12}^2, the range of m_{23}^2 is determined by the relative orientation of $\mathbf{k}_2$ and $\mathbf{k}_3$, that is whether they are parallel or antiparallel:

$$\begin{aligned} (m_{23}^2)_{\mathrm{min}} &= (E^{(2)}_{\mathbf{k}_2} + E^{(3)}_{\mathbf{k}_3})^2 - (|\mathbf{k}_2| + |\mathbf{k}_3|)^2 \\ &= (E^{(2)}_{\mathbf{k}_2} + E^{(3)}_{\mathbf{k}_3})^2 - \left(\sqrt{(E^{(2)}_{\mathbf{k}_2})^2 - m_2^2} + \sqrt{(E^{(3)}_{\mathbf{k}_3})^2 - m_3^2}\right)^2, & (4.33a)\\ (m_{23}^2)_{\mathrm{max}} &= (E^{(2)}_{\mathbf{k}_2} + E^{(3)}_{\mathbf{k}_3})^2 - (|\mathbf{k}_2| - |\mathbf{k}_3|)^2 \end{aligned}$$

$$= (E^{(2)}_{\mathbf{k}_2} + E^{(3)}_{\mathbf{k}_3})^2 - \left(\sqrt{(E^{(2)}_{\mathbf{k}_2})^2 - m_2^2} - \sqrt{(E^{(3)}_{\mathbf{k}_3})^2 - m_3^2}\right)^2 . \tag{4.33b}$$

This can be expressed in terms of m_{12}^2 when switching to the m_{12} rest system (see [1]),

$$(m_{23}^2)_{\min} = (\widetilde{E}^{(2)}_{\mathbf{k}_2} + \widetilde{E}^{(3)}_{\mathbf{k}_3})^2 - \left(\sqrt{(\widetilde{E}^{(2)}_{\mathbf{k}_2})^2 - m_2^2} + \sqrt{(\widetilde{E}^{(3)}_{\mathbf{k}_3})^2 - m_3^2}\right)^2 , \tag{4.34a}$$

$$(m_{23}^2)_{\max} = (\widetilde{E}^{(2)}_{\mathbf{k}_2} + \widetilde{E}^{(3)}_{\mathbf{k}_3})^2 - \left(\sqrt{(\widetilde{E}^{(2)}_{\mathbf{k}_2})^2 - m_2^2} - \sqrt{(\widetilde{E}^{(3)}_{\mathbf{k}_3})^2 - m_3^2}\right)^2 , \tag{4.34b}$$

with the new energies

$$E^{(2)}_{\mathbf{k}_2} \to \widetilde{E}^{(2)}_{\mathbf{k}_2} = \frac{1}{2\sqrt{m_{12}^2}}(m_{12}^2 - m_1^2 + m_2^2), \tag{4.35a}$$

$$E^{(3)}_{\mathbf{k}_3} \to \widetilde{E}^{(3)}_{\mathbf{k}_3} = \frac{1}{2\sqrt{m_{12}^2}}(M^2 - m_{12}^2 - m_3^2). \tag{4.35b}$$

4.2 Determining the isospin factor $\mathcal{I}$

When calculating decays into particles with isospin $\neq 0$, the isospin in the outgoing channel has to be considered. A neutral meson decaying into two kaons, could for example decay into a K^+K^- pair or a $K^0\bar{K}^0$ pair. In order to account for such cases, an isospin factor $\mathcal{I}$ is introduced,

$$\Gamma \to \mathcal{I} \cdot \Gamma, \tag{4.36}$$

which in the above example takes on the value $\mathcal{I} = 2$.

4.3 Explicit Calculation of Two-Body Decays

In order to calculate the decay width in eqs. (4.14) and (4.31), one needs the decay amplitude $-i\mathcal{M}$, which is typically a function of certain parameters appearing in the Lagrangian.
If there appears a derivative coupling, it is replaced by the particles momentum $\pm ik^\mu$ ($-$ for an incoming particle, $+$ for an outgoing particle.)
Some useful relations between the particles' momenta read (here a particle with momentum k_0 decays into two particles with momenta k_1 and k_2, so that $k_0^\mu = k_1^\mu + k_2^\mu$),

$$k_1 \cdot k_2 = \tfrac{1}{2}(m_0^2 - m_1^2 - m_2^2),$$
$$k_0 \cdot k_1 = \tfrac{1}{2}(m_0^2 + m_1^2 - m_2^2),$$
$$k_0 \cdot k_2 = \tfrac{1}{2}(m_0^2 - m_1^2 + m_2^2).$$

For an incoming (axial-)vector particle with polarization $\epsilon^{(\alpha)}$, one has to take the mean value of all possible polarizations, whereas for an outgoing (axial-)vector, the sum over all polarizations is performed. A useful relation for the sum over polarizations is

$$\sum_\alpha \epsilon^{(\alpha)}_\mu(k)\epsilon^{(\alpha)}_\nu(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2},$$

where $g_{\mu\nu}$ is the Minkowski-metric and m is the particle's mass.

4.3.1 Decaying spin-0 state: $S \to PP$

The fields and their momenta are denoted by

field	symbol	mass	derivative coupling
incoming spin-0 state	S	m_S	$-\mathrm{i}k_S^\mu$
outgoing spin-0 state	P_1	m_{P_1}	$+\mathrm{i}k_{P_1}^\mu$
outgoing spin-0 state	P_2	m_{P_2}	$+\mathrm{i}k_{P_2}^\mu$

TABLE 4.1: Notation for a $S \to PP$ decay.

The most general form of an interaction Lagrangian, resulting from this model's Lagrangian eq. (3.17) reads

$$\begin{aligned}\mathcal{L}_{S\to PP} = A\,SP_1P_2 + B\,S(\partial_\mu P_1)(\partial^\mu P_2)\\ + C\,(\partial_\mu S)(\partial^\mu P_1)P_2 + D\,(\partial_\mu S)P_1(\partial^\mu P_2),\end{aligned} \tag{4.37}$$

where $A, B, C, D \in \mathbb{C}$ are functions of the parameters entering our Lagrangian. The decay amplitude is now calculated as

$$-\mathrm{i}\mathcal{M}_{S\to PP} = \mathrm{i}g \tag{4.38a}$$

$$= \mathrm{i}\big[A + B(\mathrm{i}^2)k_{P_1\mu}k_{P_2}^\mu + C(-\mathrm{i}^2)k_{S\mu}k_{P_1}^\mu + D(-\mathrm{i}^2)k_{S\mu}k_{P_2}^\mu\big] \tag{4.38b}$$

$$\begin{aligned}= \mathrm{i}\big[A - \tfrac{1}{2}B(m_S^2 - m_{P_1}^2 - m_{P_2}^2) + \tfrac{1}{2}C(m_S^2 + m_{P_1}^2 - m_{P_2}^2)\\ + \tfrac{1}{2}D(m_S^2 - m_{P_1}^2 + m_{P_2}^2)\big],\end{aligned} \tag{4.38c}$$

and the decay width reads

$$\Gamma_{S\to PP} = \mathcal{I}\frac{k_f}{8\pi m_S^2}\big|-\mathrm{i}\mathcal{M}_{S\to PP}\big|^2. \tag{4.39}$$

4.3.2 Decaying spin-0 state: $S \to AP$

The fields and their momenta are denoted by

field	symbol	mass	derivative coupling	polarization
incoming spin-0 state	S	m_S	$-\mathrm{i}k_S^\mu$	
outgoing spin-1 state	A	m_A	$+\mathrm{i}k_A^\mu$	$\epsilon_\mu^{(\alpha)}(k_A)$
outgoing spin-1 state	P	m_P	$+\mathrm{i}k_P^\mu$	

TABLE 4.2: Notation for a $S \to AP$ decay.

The most general form of an interaction Lagrangian, resulting from this model's Lagrangian eq. (3.17) reads

$$\mathcal{L}_{S\to AP} = A\,SA_\mu(\partial^\mu P) + B\,(\partial^\mu S)A_\mu P, \tag{4.40}$$

where $A, B \in \mathbb{C}$ are functions of the parameters entering our Lagrangian. The decay amplitude is now calculated as

$$-\mathrm{i}\mathcal{M}^{(\alpha)}_{S\to AP} = \mathrm{i}\left[A\epsilon_\mu^{(\alpha)}(k_A)\mathrm{i}k_P^\mu + B(-\mathrm{i}k_S^\mu)\epsilon_\mu^{(\alpha)}(k_A)\right] \tag{4.41a}$$

$$= \epsilon_\mu^{(\alpha)}(k_A)\left[-Ak_P^\mu + Bk_S^\mu\right] \tag{4.41b}$$

$$\equiv \epsilon_\mu^{(\alpha)}(k_A)h^\mu. \tag{4.41c}$$

Now summing over all polarizations yields

$$\begin{aligned}
|-\mathrm{i}\mathcal{M}_{S\to AP}|^2 &= \sum_\alpha \left|-\mathrm{i}\mathcal{M}^{(\alpha)}_{S\to AP}\right|^2 && (4.42a)\\
&= \sum_\alpha \epsilon_\mu^{(\alpha)}(k_A)h^{*\mu}\epsilon_\nu^{(\alpha)}(k_A)h^\nu && (4.42b)\\
&= h^{*\mu}h^\nu\left[-g_{\mu\nu} + \frac{k_{A\mu}k_{A\nu}}{m_A^2}\right] && (4.42c)\\
&= -\left|h^\mu\right|^2 + \frac{\left|h^\mu k_{A\mu}\right|^2}{m_A^2} && (4.42d)\\
&= -|A|^2(k_P^\mu)^2 + (A^*B + B^*A)k_P\cdot k_S - |B|^2(k_s^\mu)^2\\
&\quad + \frac{1}{m_A^2}\left[|A|^2(k_P\cdot k_A)^2 - (A^*B+B^*A)(k_P\cdot k_A)(k_S\cdot k_A)\right.\\
&\quad \left.+ |B|^2(k_S\cdot k_A)^2\right] && (4.42e)\\
&= -|A|^2m_P^2 + \frac{1}{2}(A^*B+B^*A)(m_S^2+m_P^2-m_A^2) - |B|^2m_S^2\\
&\quad + \frac{|A|^2}{4m_A^2}(m_S^2-m_A^2-m_P^2)^2 - \frac{(A^*B+B^*A)}{4m_A^2}(m_S^2-m_A^2-m_P^2)\\
&\quad \times (m_S^2+m_A^2-m_P^2) + \frac{|B|^2}{4m_A^2}(m_S^2+m_A^2-m_P^2)^2 && (4.42f)\\
&= (A-B)(A^*-B^*)\frac{1}{4m_A^2}\left[m_A^4 + (m_P^2-m_S^2)^2 - 2m_A^2(m_P^2+m_S^2)\right], && (4.42g)
\end{aligned}$$

and the decay width reads

$$\Gamma_{S\to AP} = \mathcal{I}\frac{k_f}{8\pi m_S^2}\left|-\mathrm{i}\mathcal{M}_{S\to AP}\right|^2. \tag{4.43}$$

4.3.3 Decaying spin-0 state: $S \to VV$

The fields and their momenta are denoted by

field	symbol	mass	derivative coupling	polarization
incoming spin-0 state	S	m_S	$-\mathrm{i}k_S^\mu$	
outgoing spin-1 state	V_1	m_{V_1}	$+\mathrm{i}k_{V_1}^\mu$	$\epsilon_\mu^{(\alpha)}(k_{V_1})$
outgoing spin-1 state	V_2	m_{V_2}	$+\mathrm{i}k_{V_2}^\mu$	$\epsilon_\mu^{(\alpha)}(k_{V_2})$

TABLE 4.3: Notation for a $S \to VV$ decay.

The most general form of an interaction Lagrangian, resulting from this model's Lagrangian eq. (3.17) reads

$$\mathcal{L}_{S\to VV} = A\, S V_{1\mu} V_2^\mu, \tag{4.44}$$

where $A \in \mathbb{C}$ is a function of the parameters entering our Lagrangian. The decay amplitude is now calculated as

$$-\mathrm{i}\mathcal{M}^{(\alpha\beta)}_{S\to VV} = \mathrm{i}\left[A\epsilon_\mu^{(\alpha)}(k_{V_1})\epsilon^{\mu(\beta)}(k_{V_2})\right] \tag{4.45}$$

$$= \mathrm{i}\epsilon_\mu^{(\alpha)}(k_{V_1})\epsilon_\nu^{(\beta)}(k_{V_2})[Ag^{\mu\nu}] \equiv \epsilon_\mu^{(\alpha)}(k_{V_1})\epsilon_\nu^{(\beta)}(k_{V_2})h^{\mu\nu}, \tag{4.46}$$

with outgoing vectors, so summing over polarizations yields

$$\begin{aligned}
\left|-\mathrm{i}\mathcal{M}_{S\to VV}\right|^2 &= \sum_{\alpha\beta}\left|-\mathrm{i}\mathcal{M}_{S\to VV}^{(\alpha\beta)}\right|^2 \\
&= \sum_{\alpha\beta}\epsilon_\mu^{(\alpha)}(k_{V_1})\epsilon_\nu^{(\beta)}(k_{V_2})h^{*\mu\nu}\epsilon_\kappa^{(\alpha)}(k_{V_1})\epsilon_\lambda^{(\beta)}(k_{V_2})h^{\kappa\lambda} \\
&= h^{*\mu\nu}h^{\kappa\lambda}\left[-g_{\mu\kappa}+\frac{k_{V_1\mu}k_{V_1\kappa}}{m_{V_1}^2}\right]\left[-g_{\nu\lambda}+\frac{k_{V_2\nu}k_{V_2\lambda}}{m_{V_2}^2}\right] \\
&= h^{*\mu\nu}h^{\kappa\lambda}\left[g_{\mu\kappa}g_{\nu\lambda}-g_{\mu\kappa}\frac{k_{V_2\nu}k_{V_2\lambda}}{m_{V_2}^2}-g_{\nu\lambda}\frac{k_{V_1\mu}k_{V_1\kappa}}{m_{V_1}^2}+\frac{k_{V_1\mu}k_{V_1\kappa}k_{V_2\nu}k_{V_2\lambda}}{m_{V_1}^2m_{V_2}^2}\right] \\
&= |h^{\mu\nu}|^2-\frac{|h^{\mu\nu}k_{V_2\nu}|^2}{m_{V_2}^2}-\frac{|h^{\mu\nu}k_{V_1\nu}|}{m_{V_1}^2}+\frac{|h^{\mu\nu}k_{V_1\mu}k_{V_2\nu}|}{m_{V_1}^2m_{V_2}^2} \\
&= |A|^2\left[4-\frac{|k_{V_2}^\mu|^2}{m_{V_2}^2}-\frac{|k_{V_1}^\mu|}{m_{V_1}^2}+\frac{(k_{V_1}\cdot k_{V_2})^2}{m_{V_1}^2m_{V_2}^2}\right] \\
&= |A|^2\left[2+\frac{(m_S^2-m_{V_1}^2-m_{V_2}^2)^2}{4m_{V_1}^2m_{V_2}^2}\right],
\end{aligned} \tag{4.47}$$

and the decay width reads

$$\Gamma_{S\to VV} = \mathcal{I}\frac{k_f}{8\pi m_S^2}\left|-\mathrm{i}\mathcal{M}_{S\to VV}\right|^2. \tag{4.48}$$

4.4 Explicit Calculation of Three-Body Decays

In a three-body decay, the momenta of the resulting particles can take on different values within certain boundaries, as imposed by energy-momentum conservation. Therefore, one has to integrate over the new variables $\mathrm{d}\left(m_{12}^2\right)$ and $\mathrm{d}\left(m_{23}^2\right)$, as shown in section 4.1.2. With derivative couplings in the Lagrangian density, there will be Minkowski products of different momenta. Some useful relations for simplifying those expressions are listed below, in order to express everything via m_{12}^2 and m_{23}^2:

$$(k_1^\mu+k_2^\mu)^2 \equiv m_{12}^2 \quad\leftrightarrow\quad k_1\cdot k_2 = \frac{1}{2}(m_{12}^2-m_1^2-m_2^2), \tag{4.49a}$$

$$(k_2^\mu+k_3^\mu)^2 \equiv m_{23}^2 \quad\leftrightarrow\quad k_2\cdot k_3 = \frac{1}{2}(m_{23}^2-m_2^2-m_3^2), \tag{4.49b}$$

$$\begin{aligned}(k_1^\mu+k_3^\mu)^2 \equiv m_{13}^2 \quad\leftrightarrow\quad k_1\cdot k_3 &= \frac{1}{2}(m_{13}^2-m_1^2-m_3^2) \\ &= \frac{1}{2}(M^2+m_2^2-m_{12}^2-m_{23}^2),\end{aligned} \tag{4.49c}$$

where in the last equation $m_{12}^2+m_{23}^2+m_{13}^2 = M^2+m_1^2+m_2^2+m_3^2$ was used.

4.4.1 Decaying scalar state $S \to SSS$

This will be the only relevant three-body decay in this work, since on the one hand, excited higher-spin mesons are not present in this model, and on the other hand,

would-be outgoing vector mesons are too heavy for the decay to be kinematically allowed. Furthermore, possible other values for the parity (i.e. considering pseudoscalar mesons instead) do not alter the calculations below.
The fields and their momenta are denoted by

field	symbol	mass	derivative coupling
incoming scalar	S_0	m_0	$-\mathrm{i}k_0^\mu$
outgoing scalar	S_1	m_1	$+\mathrm{i}k_1^\mu$
outgoing scalar	S_2	m_2	$+\mathrm{i}k_2^\mu$
outgoing scalar	S_3	m_3	$+\mathrm{i}k_3^\mu$

Table 4.4: Notation for a $S \to SSS$ decay.

The most general form of an interaction Lagrangian for a $S \to SSS$ decay (with no derivative coupling of the decaying scalar, since this does not occur in this work) is given by

$$\begin{aligned}\mathcal{L}_I &= A\, S_0 S_1 S_2 S_3 + B S_0 (\partial_\mu S_1)(\partial^\mu S_2) S_3 \\ &\quad + C\, S_0 (\partial_\mu S_1) S_2 (\partial^\mu S_3) + D\, S_0 S_1 (\partial_\mu S_2)(\partial^\mu S_3),\end{aligned} \tag{4.50}$$

where A, B, C, $D \in \mathbb{C}$ are constant coefficients. The decay amplitude reads

$$-\mathrm{i}\mathcal{M}_{S\to SSS} = \mathrm{i}g \tag{4.51a}$$

$$= \mathrm{i}\Big[A + B(\mathrm{i}k_1^\mu)(\mathrm{i}k_{2\mu}) + C(\mathrm{i}k_1^\mu)(\mathrm{i}k_{3\mu}) + D(\mathrm{i}k_2^\mu)(\mathrm{i}k_{3\mu})\Big] \tag{4.51b}$$

$$\begin{aligned}&= \mathrm{i}\Big[A - \frac{B}{2}(m_{12}^2 - m_1^2 - m_2^2) - \frac{C}{2}(M^2 + m_2^2 - m_{12}^2 - m_{23}^2) \\ &\quad - \frac{D}{2}(m_{23}^2 - m_2^2 - m_3^2)\Big],\end{aligned} \tag{4.51c}$$

and the decay width is calculated with eq. (4.31)

$$\Gamma_{S\to SSS} = \frac{1}{(2\pi)^3 32 M^3} \int |-\mathrm{i}\mathcal{M}_{S\to SSS}|^2 \,\mathrm{d}\big(m_{12}^2\big)\,\mathrm{d}\big(m_{23}^2\big)\,. \tag{4.52}$$

Chapter 5

Results

5.1 Mass Terms of Excited Mesons

We obtained the mass terms in analogy to a general Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 - V(\varphi). \tag{5.1}$$

For the scalar and pseudo-scalar mesons, this yields

$$m^2_{\sigma_N^E} = m_0^{\star 2} + \frac{\phi_N^2}{2}(4\kappa_2 + \lambda_1^\star + \lambda_2^\star + \xi_2) + \frac{1}{2}\lambda_1^\star\phi_S^2, \tag{5.2a}$$

$$m^2_{\sigma_S^E} = m_0^{\star 2} - 2\delta + \frac{1}{2}\lambda_1^\star\phi_N^2 + \frac{\phi_S^2}{2}(4\kappa_2 + \lambda_1^\star + 2\lambda_2^\star + 2\xi_2), \tag{5.2b}$$

$$m^2_{\eta_N^E} = m_0^{\star 2} + \frac{\phi_N^2}{2}(\lambda_1^\star + \lambda_2^\star - \xi_2) + \frac{\phi_S^2}{2}\lambda_1^\star, \tag{5.2c}$$

$$m^2_{\eta_S^E} = m_0^{\star 2} - 2\delta + \frac{\phi_N^2}{2}\lambda_1^\star + \frac{\phi_S^2}{2}(\lambda_1^\star + 2\lambda_2^\star - 2\xi_2), \tag{5.2d}$$

$$m^2_{a_0^E} = m_0^{\star 2} + \frac{\phi_N^2}{2}(\lambda_1^\star + \lambda_2^\star + \xi_2) + \frac{\phi_S^2}{2}\lambda_1^\star, \tag{5.2e}$$

$$m^2_{\pi^E} = m_0^{\star 2} + \frac{\phi_N^2}{2}(\lambda_1^\star + \lambda_2^\star - \xi_2) + \frac{\phi_S^2}{2}\lambda_1^\star, \tag{5.2f}$$

$$m^2_{K_S^E} = m_0^{\star 2} - \delta + \frac{\phi_N^2}{4}(2\lambda_1^\star + \lambda_2^\star) + \frac{\phi_N\phi_S}{\sqrt{2}}\xi_2 + \frac{\phi_S^2}{2}(\lambda_1^\star + \lambda_2^\star), \tag{5.2g}$$

$$m^2_{K^E} = m_0^{\star 2} - \delta + \frac{\phi_N^2}{4}(2\lambda_1^\star + \lambda_2^\star) - \frac{\phi_N\phi_S}{\sqrt{2}}\xi_2 + \frac{\phi_S^2}{2}(\lambda_1^\star + \lambda_2^\star). \tag{5.2h}$$

Some parameters in eq. (5.2) appear in the same fashion, so that some equations are linearly dependent on each other. Identifying these terms and rewriting yields

$$m^2_{\sigma_N^E} = C_1 + \frac{1}{2}\xi_2\phi_N^2, \tag{5.3a}$$

$$m^2_{\sigma_S^E} = C_1 + 2C_2 + \xi_2\phi_S^2, \tag{5.3b}$$

$$m^2_{\eta_N^E} = C_1 - \frac{1}{2}\xi_2\phi_N^2, \tag{5.3c}$$

$$m^2_{\eta_S^E} = C_1 + 2C_2 - \xi_2\phi_S^2, \tag{5.3d}$$

$$m^2_{a_0^E} = C_1 + \frac{1}{2}\xi_2\phi_N^2, \tag{5.3e}$$

$$m_{\pi^E}^2 = C_1 - \frac{1}{2}\xi_2\phi_N^2, \tag{5.3f}$$

$$m_{K_S^E}^2 = C_1 + C_2 + \frac{1}{\sqrt{2}}\xi_2\phi_N\phi_S, \tag{5.3g}$$

$$m_{K^E}^2 = C_1 + C_2 - \frac{1}{\sqrt{2}}\xi_2\phi_N\phi_S, \tag{5.3h}$$

where we set $\kappa = 0$ because of its N_C dependency in the limit of a large number of colors [3] and defined the new parameters

$$C_1 = m_0^{\star 2} + \frac{1}{2}\lambda_1^\star(\phi_N^2 + \phi_S^2) + \frac{1}{2}\lambda_2^\star\phi_N^2, \qquad C_2 = \frac{1}{2}\lambda_2^\star(\phi_S^2 - \frac{1}{2}\phi_N^2) - \delta. \tag{5.4}$$

The parameters are calculated using a χ^2 fit in order to be able to calculate errors. The following experimental data is used as input (mass given in MeV)

$$m_{\sigma_N^E} = 1790 \pm 35, \tag{5.5a}$$

$$m_{\eta_N^E} = 1294 \pm 4, \tag{5.5b}$$

$$m_{\eta_S^E} = 1432 \pm 10. \tag{5.5c}$$

The reason to use these numbers is the assignment of our theoretical states σ_N^E, η_N^E and η_S^E to the resonances $f_0(1790)$, $\eta(1295)$ and $\eta(1440)$, known from the experimental data analyzed in [1,4–6]. χ^2 leads to the values (see appendix A.1 for a brief review on how to calculate errors in a χ^2 fit)

$$C_1 = (2.4 \pm 0.6) \cdot 10^6, \tag{5.6a}$$

$$C_2 = (2.5 \pm 0.2) \cdot 10^5, \tag{5.6b}$$

$$\xi_2 = 57 \pm 5, \tag{5.6c}$$

which yields the following values for the other mesons:

$$m_{\sigma_S^E} = 1961 \pm 38, \tag{5.7a}$$

$$m_{a_0^E} = 1790 \pm 35, \tag{5.7b}$$

$$m_{\pi^E} = 1294 \pm 4, \tag{5.7c}$$

$$m_{K_S^E} = 1877 \pm 36, \tag{5.7d}$$

$$m_{K^E} = 1366 \pm 6. \tag{5.7e}$$

5.2 Decay Widths

In order to calculate and predict decay widths, we need to fix some parameters that appear in our model. In order to do so, we use a χ^2 fit, which makes it possible to also calculate errors (see appendix A.1 for details).

We will use the following experimental data [1,4–6]. The decay $\Gamma_{\sigma_N^E \to 2K}$ has been estimated using the branching ratios $BR(J/\Psi \to \phi f_0(1790) \to \phi 2\pi) = (6.2 \pm 1.4) \cdot 10^{-4}$ and $BR(J/\Psi \to \phi f_0(1790) \to \phi 2K) = (1.6 \pm 0.8) \cdot 10^{-4}$ from [5]. As mentioned in section 5.1, σ_N^E has been assigned to $f_0(1790)$, η_N^E has been assigned to $\eta(1295)$ and η_S^E to $\eta(1440)$. Then we have

$$\Gamma_{\eta_S^E \to K^* K} = 26 \pm 2.6, \tag{5.8a}$$

$$\Gamma_{\sigma_N^E \to 2\pi} = 270 \pm 45, \tag{5.8b}$$

$$\Gamma_{\sigma_N^E \to 2K} = 70 \pm 40, \tag{5.8c}$$

$$\Gamma_{\eta_N^E} = 55 \pm 5. \tag{5.8d}$$

In this preliminary study, there will be no consideration of mixing of particles that have the same quantum numbers. Furthermore, we also omit excited-state parameters that influence ground states since the Lagrangian (3.18) is sufficient for the description of the latter [3] and we disregard parameters of terms that lead to kinematically forbidden decays. If, in addition, we omit parameters that are suppressed in the limit of large number of colors then the only parameters remaining in eq. (5.8) are $h_2^\star$ and $h_3^\star$. This means, we could perform the χ^2 fit using only two experimental values as input, but also more. We tried all combinations of eq. (5.8) as input for a χ^2 fit; discarding combinations that lead to unmeasurably large decay widths, we are left with the two scenarios discussed in the next two sections.

5.2.1 Scenario 1: scalar input

For the first scenario, we chose $\Gamma_{\sigma_N^E \to 2\pi}$ and $\Gamma_{\sigma_N^E \to 2K}$ as input. Here, the optimal values for $h_2^\star$ and $h_3^\star$ result from a χ^2 fit to:

$$h_2^\star = -67 \pm 63, \tag{5.9a}$$

$$h_3^\star = -79 \pm 63. \tag{5.9b}$$

This section presents the results for the various decays that we calculated (given in MeV).

Decaying particle: σ_N^E

$$\Gamma_{\sigma_N^E \to \eta\eta} = 7 \pm 1 \qquad \Gamma_{\sigma_N^E \to \eta\eta'} = 10 \pm 2 \qquad \Gamma_{\sigma_N^E \to \pi\pi} = 270 \pm 45$$

$$\Gamma_{\sigma_N^E \to KK} = 70 \pm 40 \qquad \Gamma_{\sigma_N^E \to a_1\pi} = 47 \pm 8 \qquad \Gamma_{\sigma_N^E \to K_1 K} = 0$$

$$\Gamma_{\sigma_N^E \to f_{1N}\eta} = 1 \pm 0 \qquad \Gamma_{\sigma_N^E \to \sigma_N\pi\pi} = 0 \qquad \Gamma^{\text{tot}}_{\sigma_N^E} = 405 \pm 96$$

Decaying particle: σ_S^E

$$\Gamma_{\sigma_S^E \to \eta\eta} = 6 \pm 1 \qquad \Gamma_{\sigma_S^E \to \eta\eta'} = 12 \pm 2 \qquad \Gamma_{\sigma_S^E \to \eta'\eta'} = 1 \pm 0$$

$$\Gamma_{\sigma_S^E \to KK} = 21^{+40}_{-21} \qquad \Gamma_{\sigma_S^E \to K_1 K} = 2^{+5}_{-2} \qquad \Gamma^{\text{tot}}_{\sigma_S^E} = 42^{+48}_{-26}$$

Decaying particle: η_N^E

$$\Gamma^{\text{tot}}_{\eta_N^E} = 7 \pm 3$$

Decaying particle: η_S^E

$$\Gamma_{\eta_S^E \to K^\star K} = 128^{+204}_{-128} \qquad \Gamma_{\eta_S^E \to \pi KK} = 28^{+41}_{-28} \qquad \Gamma^{\text{tot}}_{\eta_S^E} = 156^{+245}_{-156}$$

Decaying particle: a_0^E

$$\Gamma_{a_0^E\to\pi\eta} = 72 \pm 12 \qquad \Gamma_{a_0^E\to\pi\eta'} = 32 \pm 5 \qquad \Gamma_{a_0^E\to f_{1N}\pi} = 16 \pm 3$$
$$\Gamma_{a_0^E\to KK} = 70 \pm 40 \qquad \Gamma_{a_0^E\to a_1\eta} = 1 \pm 0 \qquad \Gamma_{a_0^E\to K_1K} = 0$$
$$\Gamma_{a_0^E\to a_0\pi\pi} = 0 \qquad \Gamma^{\text{tot}}_{a_0^E} = 191 \pm 60$$

Decaying particle: π^E

The total decay width of π^E is compatible with values between 0 and $\mathcal{O}(\text{GeV})$.

Decaying particle: K_S^E

$$\Gamma_{K_S^E\to K\eta} = 4^{+7}_{-4} \qquad \Gamma_{K_S^E\to K\eta'} = 24 \pm 4 \qquad \Gamma_{K_S^E\to f_{1N}K} = 1 \pm 1$$
$$\Gamma_{K_S^E\to K\pi} = 51 \pm 35 \qquad \Gamma_{K_S^E\to a_1K} = 3 \pm 2 \qquad \Gamma_{K_S^E\to K_1\eta} = 0$$
$$\Gamma_{K_S^E\to K_1\pi} = 6 \pm 4 \qquad \Gamma_{K_S^E\to K_S\pi\pi} = 0 \qquad \Gamma^{\text{tot}}_{K_S^E} = 89^{+53}_{-50}$$

Decaying particle: K^E

The total decay width of K^E is compatible with values between 0 and $\mathcal{O}(\text{GeV})$.

5.2.2 Scenario 2: pseudoscalar input

The second scenario takes the pseudoscalar decay widths as input. Here, the optimal values for $h_2^\star$ and $h_3^\star$ result from a χ^2 fit with $\Gamma_{\eta_N^E}$ and $\Gamma_{\eta_S^E\to K^*K}$ as input:

$$h_2^\star = 70 \pm 2, \tag{5.10a}$$
$$h_3^\star = 35 \pm 2. \tag{5.10b}$$

This section presents the results for the various decays that we calculated (given in MeV) now with the parameters of scenario 2.

Decaying particle: σ_N^E

The total decay width of σ_N^E is compatible with values of $\mathcal{O}(\text{GeV})$.

Decaying particle: σ_S^E

The total decay width of σ_S^E is compatible with values of $\mathcal{O}(\text{GeV})$.

Decaying particle: η_N^E

$$\Gamma_{\eta_N^E} = 55 \pm 5 = \Gamma^{\text{tot}}_{\eta_N^E}$$

Decaying particle: η_S^E

$$\Gamma_{\eta_S^E\to K*K} = 26 \pm 3 \qquad \Gamma_{\eta_S^E\to\pi KK} = 3 \pm 0 \qquad \Gamma^{\text{tot}}_{\eta_S^E} = 29 \pm 3$$

Decaying particle: a_0^E

The total decay width of a_0^E is compatible with values of $\mathcal{O}(\text{GeV})$.

Decaying particle: π^E

$$\Gamma_{\pi^E\to\rho\pi} = 368 \pm 37 \qquad \Gamma_{\pi^E\to\pi\pi\pi} = 204 \pm 15$$
$$\Gamma_{\pi^E\to\pi KK} = 2 \pm 0 \qquad \Gamma^{\text{tot}}_{\pi^E} = 574 \pm 52$$

Decaying particle: K_S^E

The total decay width of K_S^E is compatible with values of $\mathcal{O}(\text{GeV})$.

Decaying particle: K^E

$$\Gamma_{K^E\to K^\star\pi} = 112 \pm 11 \qquad \Gamma_{K^E\to\omega_N K} = 7 \pm 1 \qquad \Gamma_{K^E\to\rho K} = 20 \pm 2$$
$$\Gamma_{K^E\to K\pi\pi} = 35 \pm 4 \qquad \Gamma_{K^E\to K\pi\eta} = 0 \qquad \Gamma^{\text{tot}}_{K^E} = 174 \pm 18$$

5.2.3 Discussion

Scenario 1 used the decay widths $\Gamma_{\sigma_N^E\to 2\pi}$ and $\Gamma_{\sigma_N^E\to 2K}$ as input. Here, we assume the resonance $f_0(1790)$ to be an excited $\bar{q}q$ state and calculate a total decay width of $\sim (400 \pm 100)$ MeV. We were able to present predictions for other decay widths, however, the excited pseudoscalar mesons' decay widths are compatible with values between 0 and $\mathcal{O}(\text{GeV})$. Scenario 2 took $\Gamma_{\eta_N^E}$ and $\Gamma_{\eta_S^E\to K^*K}$ as input for the χ^2 fit. There, we assume the resonances $\eta(1440)$ and $\eta(1295)$ to be excited $\bar{q}q$ states. However in this case, the decaying scalar mesons came with unmeasurably large decay widths.
In the second scenario, we assumed the decay width of η_N^E to consist mostly of the three decays $\eta_N^E \to \eta\pi\pi$, $\eta_N^E \to \eta'\pi\pi$ and $\eta_N^E \to \pi KK$. Since, due to experimental uncertainties, we cannot say for sure that these three decays represent the full decay width, we prefer scenario 1.
A consequence of the scenario is that the masses and decay widths of our scalar isotriplet and isodoublet correspond very well to the experimental data on the resonances $a_0(1950)$ and $K_0^\star(1950)$, both of which can be found in listings of Ref. [1]. That means that $a_0(1950)$ and $K_0^\star(1950)$ are preferred to represent excited $\bar{q}q$ states in our model. Interestingly, our scalar $\bar{s}s$ state turns out to be much narrower than any state of similar mass listed in Ref. [1]. Such states may contain admixture of non-$\bar{q}q$ structures (such as for examples glueballs, bound states of gluons) but that would go beyond this study.

Chapter 6

Summary and Outlook

Quantum Chromodynamics (QCD) is the theory of strong interaction whose degrees of freedom (quarks and gluons) are not of direct relevance in the low-energy region ($\simeq \mathcal{O}(1\text{ GeV})$). For that reason, models are used; the difference is that when using a model, we take composite particles, mesons, as degrees of freedom. We have used the Linear Sigma Model whose main features are appropriate handling of all of QCD's symmetries, as well as their breaking. We have tested whether certain measured resonances can be interpreted as $\bar{q}q$ states by using their available experimental data as input to fix our model's parameters. Thereby we were able to make predictions concerning other mesons' decay widths. We have two scenarios but prefer one of them because the other scenario assumes a total decay width to consist of (only) three separate decays, which we cannot say for sure. Then, according to our results, the known resonances $f_0(1790)$, $a_0(1950)$ and $K_0^\star(1950)$ – see Ref. [1] – are very likely to represent excited $\bar{q}q$ states.

Possible extensions of this work include the addition of excited spin 1 mesons (vector and axial-vector mesons) and their interaction with the present fields, as well as calculating more decays, that depend on other parameters than the ones fitted. Both approaches are currently limited by the scarce situation of too little experimental data for excited-state candidates. Future measurements at, e.g., the Facility for Antiproton and Ion Research (FAIR) in Germany are expected to provide new experimental information.

Appendix A

Appendix

A.1 Errors of a χ^2 fit

This section will explain how to calculate errors out of a χ^2 fit. [7]
The χ^2 function is defined as

$$\chi^2(p_1, \ldots, p_k) = \sum_{i=1}^{N} \left(\frac{O_i^{\text{th}}(p_1, \ldots, p_k) - O_i^{\text{ex}}}{\Delta O_i^{\text{ex}}} \right)^2, \tag{A.1}$$

where p_j are $k \leq N$ parameters, O_i^{th} are theoretical values for an observable (depending on the parameters p_j), O_i^{ex} are experimentally measured values and ΔO_i^{ex} their errors.
The theoretical error is given by

$$\Delta O_i^{\text{th}} = \sqrt{\sum_{j=1}^{k} \frac{1}{\lambda_j} (B\mathbf{f}_i)_j^2}. \tag{A.2}$$

Here, B is a matrix that diagonalizes the Hessian matrix that appears in a power expansion of χ^2,

$$BHB^T = \lambda = \text{diag}(\lambda_1, \ldots, \lambda_k), \tag{A.3}$$

λ_j the j-th eigenvalue of the Hessian matrix and $\mathbf{f}$ a vector, defined as

$$\mathbf{f}_i = \left. \begin{pmatrix} \frac{\partial O_i^{\text{th}}}{\partial p_1} \\ \vdots \\ \frac{\partial O_i^{\text{th}}}{\partial p_k} \end{pmatrix} \right|_{p_j = p_j^{\text{min}}}, \tag{A.4}$$

and $p_1^{\text{min}}, \ldots, p_k^{\text{min}}$ are the values that minimize χ^2.

Finally, we calculate the parameters' errors by taking the square root of the diagonal elements of the inverse Hessian matrix H^{-1}

$$\Delta p_i = \sqrt{H_{ii}^{-1}}, \quad i = 1, \ldots, k. \tag{A.5}$$

A.2 Coefficients in the Lagrangian

Decay $\sigma_N^E \to \eta_N \eta_N$

$$\mathcal{L}_I = A\,\sigma_N^E \eta_N \eta_N + B\,(\partial_\mu \sigma_N^E)(\partial^\mu \eta_N)\eta_N + C\,\sigma_N^E(\partial_\mu \eta_N)^2$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_{\eta_N}^2 \kappa_1 \phi_N - \frac{1}{4} Z_{\eta_N}^2 \xi_1 \phi_N \\
B &= g_1 w_{f_{1N}} Z_{\eta_N}^2 \alpha \\
C &= -g_1^\star w_{f_{1N}} Z_{\eta_N}^2 \alpha + \frac{1}{2} w_{f_{1N}}^2 Z_{\eta_N}^2 \phi_N (h_1^\star + h_2^\star - h_3^\star) + g_1 g_1^\star w_{f_{1N}}^2 Z_{\eta_N}^2 \alpha \phi_N
\end{aligned}$$

Decay $\sigma_N^E \to \eta_S \eta_S$

$$\mathcal{L}_I = A\,\sigma_N^E \eta_S \eta_S + B\,(\partial_\mu \sigma_N^E)(\partial^\mu \eta_S)\eta_S + C\,\sigma_N^E(\partial_\mu \eta_S)^2$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_{\eta_S}^2 \kappa_1 \phi_N \\
B &= 0 \\
C &= \frac{1}{2} h_1^\star w_{f_{1S}}^2 Z_{\eta_S}^2 \phi_N
\end{aligned}$$

Decay $\sigma_N^E \to \pi\pi$

$$\mathcal{L}_I = A\,\sigma_N^E \boldsymbol{\pi}\boldsymbol{\pi} + B\,\sigma_N^E(\partial_\mu \boldsymbol{\pi})^2 + C\,(\partial_\mu \sigma_N^E)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi}$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_\pi^2 \kappa_1 \phi_N - \frac{1}{4} Z_\pi^2 \xi_1 \phi_N \\
B &= -g_1^\star w_{a_1} Z_\pi^2 \alpha + \frac{1}{2} w_{a_1}^2 Z_\pi^2 \phi_N (h_1^\star + h_2^\star - h_3^\star) + g_1 g_1^\star w_{a_1}^2 Z_\pi^2 \alpha \phi_N \\
C &= g_1 w_{a_1} Z_\pi^2 \alpha
\end{aligned}$$

Decay $\sigma_N^E \to \pi\pi^E$

$$\mathcal{L}_I = A\,\sigma_N^E \boldsymbol{\pi}\boldsymbol{\pi}^E + B\,(\partial_\mu \sigma_N^E)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi}^E + B\,(\partial_\mu \sigma_N^E)\boldsymbol{\pi}(\partial^\mu \boldsymbol{\pi}^E) + D\,\sigma_N^E(\partial_\mu \boldsymbol{\pi})(\partial^\mu \boldsymbol{\pi}^E)$$

$$\begin{aligned}
A &= -2 Z_\pi \kappa_2 \phi_N - Z_\pi \xi_2 \phi_N \\
B &= g_1^\star w_{a_1} Z_\pi \\
C &= 0 \\
D &= -g_1^\star w_{a_1} Z_\pi
\end{aligned}$$

Decay $\sigma_N^E \to KK$

$$\mathcal{L}_I = A\,\sigma_N^E KK + B\,\sigma_N^E (\partial_\mu K)^2 + C\,(\partial_\mu \sigma_N^E)(\partial^\mu K)K$$

$$\begin{aligned}
A &= -Z_K^2 \kappa_1 \phi_N - \frac{1}{2} Z_K^2 \xi_1 \phi_N + \frac{1}{2\sqrt{2}} Z_K^2 \xi_1 \phi_S \\
B &= -g_1^\star w_{K_1} Z_K^2 \alpha + w_{K_1}^2 Z_K^2 \phi_N (h_1^\star + \frac{1}{2} h_2^\star) + \frac{1}{2} g_1 g_1^\star w_{K_1}^2 Z_K^2 \alpha \phi_N - \frac{1}{\sqrt{2}} h_3^\star w_{K_1}^2 Z_K^2 \phi_S \\
&\quad + \frac{1}{\sqrt{2}} g_1 g_1^\star w_{K_1}^2 Z_K^2 \alpha \phi_S \\
C &= \frac{1}{2} g_1 w_{K_1} Z_K^2 \alpha
\end{aligned}$$

Decay $\sigma_N^E \to \omega_N \omega_N$

$$\mathcal{L}_I = A\,\sigma_N^E \omega_N^\mu \omega_{N\mu}$$

$$A = \frac{1}{2}\phi_N (h_1^\star + h_2^\star + h_3^\star)$$

Decay $\sigma_N^E \to \rho\rho$

$$\mathcal{L}_I = A\,\sigma_N^E \boldsymbol{\rho}^\mu \boldsymbol{\rho}_\mu$$

$$A = \frac{1}{2}\phi_N (h_1^\star + h_2^\star + h_3^\star)$$

Decay $\sigma_N^E \to K^\star K^\star$

$$\mathcal{L}_I = A\,\sigma_N^E K^{\star\mu} K_\mu^\star$$

$$A = (h_1^\star + \frac{1}{2} h_2^\star)\phi_N + \frac{1}{2} g_1 g_1^\star \alpha \phi_N + \frac{1}{\sqrt{2}} h_3^\star \phi_S - \frac{1}{\sqrt{2}} g_1 g_1^\star \alpha \phi_S$$

Decay $\sigma_N^E \to a_1 \pi$

$$\mathcal{L}_I = A\,\sigma_N^E \mathbf{a}_1^\mu \partial_\mu \boldsymbol{\pi} + B\,(\partial_\mu \sigma_N^E)\mathbf{a}_1^\mu \boldsymbol{\pi}$$

$$\begin{aligned}
A &= -g_1^\star Z_\pi \alpha + w_{a_1} Z_\pi \phi_N (h_1^\star + h_2^\star - h_3^\star) + 2 g_1 g_1^\star w_{a_1} Z_\pi \alpha \phi_N \\
B &= g_1 Z_\pi \alpha
\end{aligned}$$

Decay $\sigma_N^E \to K_1 K$

$$\mathcal{L}_I = A\,\sigma_N^E K_1^\mu \partial_\mu K + B\,(\partial_\mu \sigma_N^E) K_1^\mu K$$

$$\begin{aligned}
A &= -\frac{1}{2} g_1^\star Z_K \alpha + w_{K_1} Z_K \phi_N (h_1^\star + \frac{1}{2} h_2^\star) + \\
&\quad + \frac{1}{2} g_1 g_1^\star w_{K_1} Z_K \alpha \phi_N - \frac{1}{\sqrt{2}} h_3^\star w_{K_1} Z_K \phi_S + \frac{1}{\sqrt{2}} g_1 g_1^\star w_{K_1} Z_K \alpha \phi_S \\
B &= \frac{1}{2} g_1 Z_K \alpha
\end{aligned}$$

Decay $\sigma_N^E \to f_{1N} \eta_N$

$$\mathcal{L}_I = A\,\sigma_N^E f_{1N}^\mu \partial_\mu \eta_N + B\,(\partial_\mu \sigma_N^E) f_{1N}^\mu \eta_N$$

$$\begin{aligned}
A &= Z_{\eta_N} (h_1^\star + h_2^\star - h_3^\star) w_{f_{1N}} \phi_N + Z_{\eta_N} \alpha g_1 (-1 + 2 g_1 w_{f_{1N}} \phi_N) \\
B &= \alpha g_1 Z_{\eta_N}
\end{aligned}$$

Decay $\sigma_S^E \to \eta_N \eta_N$

$$\mathcal{L}_I = A\,\sigma_S^E \eta_N \eta_N + B\,(\partial_\mu \sigma_S^E)(\partial^\mu \eta_N)\eta_N + C\,\sigma_S^E (\partial_\mu \eta_N)^2$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_{\eta_N}^2 \kappa_1 \phi_S \\
B &= 0 \\
C &= \frac{1}{2} h_1^\star w_{f_{1N}}^2 Z_{\eta_N}^2 \phi_S
\end{aligned}$$

Decay $\sigma_S^E \to \eta_S \eta_S$

$$\mathcal{L}_I = A\,\sigma_S^E \eta_S \eta_S + B\,(\partial_\mu \sigma_S^E)(\partial^\mu \eta_S)\eta_S + C\,\sigma_S^E (\partial_\mu \eta_S)^2$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_{\eta_S}^2 \kappa_1 \phi_S - \frac{1}{2} Z_{\eta_S}^2 \xi_1 \phi_S \\
B &= \sqrt{2} g_1 w_{f_{1S}} Z_{\eta_S}^2 \alpha \\
C &= -\sqrt{2} g_1^\star w_{f_{1S}} Z_{\eta_S} \alpha + \frac{1}{2} w_{f_{1S}}^2 Z_{\eta_S}^2 \phi_S (h_1^\star + 2h_2^\star - 2h_3^\star) + 2 g_1 g_1^\star w_{f_{1S}}^2 Z_{\eta_S}^2 \alpha \phi_S
\end{aligned}$$

Decay $\sigma_S^E \to \pi\pi$

$$\mathcal{L}_I = A\,\sigma_S^E \boldsymbol{\pi}\boldsymbol{\pi} + B\,\sigma_S^E (\partial_\mu \boldsymbol{\pi})^2 + C\,(\partial_\mu \sigma_S^E)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi}$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_\pi^2 \kappa_1 \phi_S \\
B &= \frac{1}{2} h_1^\star w_{a_1}^2 Z_\pi^2 \phi_S \\
C &= 0
\end{aligned}$$

Decay $\sigma_S^E \to KK$

$$\mathcal{L}_I = A\,\sigma_S^E KK + B\,\sigma_S^E(\partial_\mu K)^2 + C\,(\partial_\mu\sigma_S^E)(\partial^\mu K)K$$

$$\begin{aligned}
A &= \frac{1}{2\sqrt{2}}Z_K^2\xi_1\phi_N - Z_K^2\kappa_1\phi_S - Z_K^2\xi_1\phi_S \\
B &= -\sqrt{2}g_1^\star w_{K_1}Z_K^2\alpha - \frac{1}{\sqrt{2}}h_3^\star w_{K_1}^2 Z_K^2\phi_N + \frac{1}{\sqrt{2}}g_1 g_1^\star w_{K_1}^2 Z_K^2\alpha\phi_N + \\
&\quad + w_{K_1}^2 Z_K^2\phi_S(h_1^\star + h_2^\star) + g_1 g_1^\star w_{K_1}^2 Z_K^2\alpha\phi_S \\
C &= \frac{1}{\sqrt{2}}g_1 w_{K_1}Z_K^2\alpha
\end{aligned}$$

Decay $\sigma_S^E \to \omega_N\omega_N$

$$\mathcal{L}_I = A\,\sigma_S^E\omega_N^\mu\omega_{N\mu}$$

$$A = \frac{1}{2}h_1^\star\phi_S$$

Decay $\sigma_S^E \to \rho\rho$

$$\mathcal{L}_I = A\,\sigma_S^E\boldsymbol{\rho}^\mu\boldsymbol{\rho}_\mu$$

$$A = \frac{1}{2}h_1^\star\phi_S$$

Decay $\sigma_S^E \to K^\star K^\star$

$$\mathcal{L}_I = A\,\sigma_S^E K^{\star\mu}K_\mu^\star$$

$$A = \frac{1}{\sqrt{2}}h_3^\star\phi_N - \frac{1}{\sqrt{2}}g_1 g_1^\star\alpha\phi_N + \phi_S(h_1^\star + h_2^\star) + g_1 g_1^\star\alpha\phi_S$$

Decay $\sigma_S^E \to a_1\pi$

$$\mathcal{L}_I = A\,\sigma_S^E\mathbf{a}_1^\mu\partial_\mu\boldsymbol{\pi} + B\,(\partial_\mu\sigma_S^E)\mathbf{a}_1^\mu\,\boldsymbol{\pi}$$

$$\begin{aligned}
A &= h_1^\star w_{a_1}Z_\pi\phi_S \\
B &= 0
\end{aligned}$$

Decay $\sigma_S^E \to K_1 K$

$$\mathcal{L}_I = A\,\sigma_S^E K_1^\mu \partial_\mu K + B\,(\partial_\mu \sigma_S^E) K_1^\mu K$$

$$\begin{aligned}
A &= -\frac{1}{\sqrt{2}} g_1^\star Z_K \alpha - \frac{1}{\sqrt{2}} h_3^\star w_{K_1} Z_K \phi_N + \frac{1}{\sqrt{2}} g_1 g_1^\star w_{K_1} Z_K \alpha \phi_N \\
&\quad + (h_1^\star + h_2^\star) w_{K_1} Z_K \phi_S + g_1 g_1^\star w_{K_1} Z_K \alpha \phi_S \\
B &= \frac{1}{\sqrt{2}} g_1 Z_K \alpha
\end{aligned}$$

Decay $\eta_S^E \to K K^\star$

$$\mathcal{L}_I = A\,\eta_S^E K_\mu^\star (\partial^\mu K) + B\,(\partial^\mu \eta_S^E) K_\mu^\star K$$

$$\begin{aligned}
A &= -\frac{\mathrm{i}}{\sqrt{2}} g_1^\star Z_K \alpha - \frac{\mathrm{i}}{\sqrt{2}} h_3^\star w_{K_1} Z_K \phi_N + \frac{\mathrm{i}}{\sqrt{2}} g_1 g_1^\star w_{K_1} Z_K \alpha \phi_N \\
B &= \frac{\mathrm{i}}{\sqrt{2}} g_1 Z_K \alpha
\end{aligned}$$

Decay $a_0^E \to \pi \eta_N$

$$\mathcal{L}_I = A\,\mathbf{a}_0^E \boldsymbol{\pi} \eta_N + B\,\mathbf{a}_0^E (\partial_\mu \boldsymbol{\pi})(\partial^\mu \eta_N) + C\,(\partial_\mu \mathbf{a}_0^E)(\partial^\mu \boldsymbol{\pi}) \eta_N + D\,(\partial_\mu \mathbf{a}_0^E) \boldsymbol{\pi} (\partial^\mu \eta_N)$$

$$\begin{aligned}
A &= -\frac{1}{2} Z_{\eta_N} Z_\pi \xi_1 \phi_N \\
B &= -g_1^\star Z_\pi \alpha (w_{a_1} + w_{f_{1N}}) Z_{\eta_N} + (h_2^\star - h_3^\star) w_{a_1} w_{f_{1N}} Z_{\eta_N} Z_\pi \phi_N + 2 g_1 g_1^\star w_{a_1} w_{f_{1N}} Z_{\eta_N} Z_\pi \alpha \phi_N \\
C &= g_1 w_{a_1} Z_{\eta_N} Z_\pi \alpha \\
D &= g_1 w_{f_{1N}} Z_{\eta_N} Z_\pi \alpha
\end{aligned}$$

Decay $a_0^E \to \rho \omega_N$

$$\mathcal{L}_I = A\,\mathbf{a}_0^E \boldsymbol{\rho}_\mu \omega_N^\mu$$

$$A = \phi_N (h_2^\star + h_3^\star)$$

Decay $a_0^E \to f_{1N} \pi$

$$\mathcal{L}_I = A\,\mathbf{a}_0^E f_{1N}^\mu (\partial_\mu \boldsymbol{\pi}) + B\,(\partial_\mu \mathbf{a}_0^E) f_{1N}^\mu \boldsymbol{\pi}$$

$$\begin{aligned}
A &= -g_1^\star Z_\pi \alpha + (h_2^\star - h_3^\star) w_{a_1} Z_\pi \phi_N + 2 g_1 g_1^\star w_{a_1} Z_\pi \alpha \phi_N \\
B &= g_1 Z_\pi \alpha
\end{aligned}$$

Decay $a_0^E \to \pi\eta_N^E$

$$\mathcal{L}_I = A\,\mathbf{a}_0^E\boldsymbol{\pi}\eta_N^E + B\,(\partial_\mu\mathbf{a}_0^E)(\partial^\mu\boldsymbol{\pi})\eta_N^E + C\,(\partial_\mu\mathbf{a}_0^E)\boldsymbol{\pi}(\partial^\mu\eta_N^E) + D\,\mathbf{a}_0^E(\partial_\mu\boldsymbol{\pi})(\partial^\mu\eta_N^E)$$

$$A = -Z_\pi\xi_2\phi_N$$
$$B = g_1^\star w_{a_1}Z_\pi$$
$$C = 0$$
$$D = -g_1^\star w_{a_1}Z_\pi$$

Decay $a_0^E \to KK$

$$\mathcal{L}_I = A\,a_0^{0E}KK + B\,a_0^{0E}(\partial_\mu K)(\partial^\mu K) + C\,(\partial_\mu a_0^{0E})(\partial^\mu K)K$$

$$A = \frac{1}{2}Z_K^2\xi_1\phi_N - \frac{1}{2\sqrt{2}}Z_K^2\xi_1\phi_S$$
$$B = g_1^\star w_{K_1}Z_K^2\alpha - \frac{1}{2}(h_2^\star + g_1g_1^\star\alpha)w_{K_1}^2Z_K^2\phi_N + \frac{1}{\sqrt{2}}(h_3^\star - g_1g_1^\star\alpha)w_{K_1}^2Z_K^2\phi_S$$
$$C = -\frac{1}{2}g_1w_{K_1}Z_K^2\alpha$$

Decay $a_0^E \to a_1\eta_N$

$$\mathcal{L}_I = A\,(\partial_\mu\mathbf{a}_0^E)\mathbf{a}_1^\mu\eta_N + B\,\mathbf{a}_0^E\mathbf{a}_1^\mu(\partial_\mu\eta_N)$$

$$A = g_1Z_{\eta_N}\alpha$$
$$B = -g_1^\star Z_{\eta_N}\alpha + w_{f_{1N}}Z_{\eta_N}\phi_N(h_2^\star - h_3^\star) + 2g_1g_1^\star w_{f_{1N}}Z_{\eta_N}\alpha\phi_N$$

Decay $a_0^E \to K^\star K^\star$

$$\mathcal{L}_I = A\,a_0^{0E}K_\mu^\star K^{\star\mu}$$

$$A = -\frac{1}{2}h_2^\star\phi_N - \frac{1}{2}g_1g_1^\star\alpha\phi_N - \frac{1}{\sqrt{2}}h_3^\star\phi_S + \frac{1}{\sqrt{2}}g_1g_1^\star\alpha\phi_S$$

Decay $a_0^E \to K_1K$

$$\mathcal{L}_I = A\,(\partial_\mu\mathbf{a}_0^E)K_1^\mu K + B\,\mathbf{a}_0^E K_1^\mu(\partial_\mu K)$$

$$A = -\frac{1}{2}g_1Z_K\alpha$$
$$B = \frac{1}{2}g_1^\star Z_K\alpha - \frac{1}{2}w_{K_1}Z_K\phi_N(h_2^\star + g_1g_1^\star\alpha) + \frac{1}{\sqrt{2}}w_{K_1}Z_K\phi_S(h_3^\star - g_1g_1^\star\alpha)$$

Decay $K_S^E \to K\eta_N$

$$\mathcal{L}_I = A\, K_S^E K \eta_N + B\, K_S^E (\partial_\mu K)(\partial^\mu \eta_N) + C\, (\partial_\mu K_S^E)(\partial^\mu K)\eta_N + D\, (\partial_\mu K_S^E) K (\partial^\mu \eta_N)$$

$$\begin{aligned}
A &= -\frac{1}{2\sqrt{2}} Z_K Z_{\eta_N} \xi_1 \phi_S \\
B &= -\frac{1}{2} g_1^\star w_{f_{1N}} Z_K Z_{\eta_N} \alpha - \frac{1}{2} g_1^\star w_{K_1} Z_K Z_{\eta_N} \alpha \\
&\quad + \frac{1}{4} h_2^\star w_{f_{1N}} w_{K_1} Z_K Z_{\eta_N} \phi_N - \frac{1}{2} h_3^\star w_{f_{1N}} w_{K_1} Z_K Z_{\eta_N} \phi_N \\
&\quad + \frac{3}{4} g_1 g_1^\star w_{f_{1N}} w_{K_1} Z_K Z_{\eta_N} \alpha \phi_N + \frac{1}{2\sqrt{2}} h_2^\star w_{f_{1N}} w_{K_1} Z_K Z_{\eta_N} \phi_S \\
&\quad + \frac{1}{2\sqrt{2}} g_1 g_1^\star w_{f_{1N}} w_{K_1} Z_K Z_{\eta_N} \alpha \phi_S \\
C &= \frac{1}{2} g_1 w_{K_1} Z_K Z_{\eta_N} \alpha \\
D &= \frac{1}{2} g_1 w_{f_{1N}} Z_K Z_{\eta_N} \alpha
\end{aligned}$$

Decay $K_S^E \to K\eta_S$

$$\mathcal{L}_I = A\, K_S^E K \eta_S + B\, K_S^E (\partial_\mu K)(\partial^\mu \eta_S) + C\, (\partial_\mu K_S^E)(\partial^\mu K)\eta_S + D\, (\partial_\mu K_S^E) K (\partial^\mu \eta_S)$$

$$\begin{aligned}
A &= -\frac{1}{2\sqrt{2}} Z_K Z_{\eta_S} \xi_1 \phi_N \\
B &= -\frac{1}{\sqrt{2}} g_1^\star Z_K Z_{\eta_S} (w_{K_1} + w_{f_{1S}})\alpha + \frac{1}{2\sqrt{2}} h_2^\star w_{f_{1S}} w_{K_1} Z_K Z_{\eta_S} \phi_N + \\
&\quad + \frac{1}{2\sqrt{2}} g_1 g_1^\star w_{f_{1S}} w_{K_1} Z_K Z_{\eta_S} \alpha \phi_N + (\frac{1}{2} h_2^\star - h_3^\star) w_{f_{1S}} w_{K_1} Z_K Z_{\eta_S} \phi_S + \\
&\quad + \frac{3}{2} g_1 g_1^\star w_{f_{1S}} w_{K_1} Z_K Z_{\eta_S} \alpha \phi_S \\
C &= \frac{1}{\sqrt{2}} g_1 w_{K_1} Z_K Z_{\eta_S} \alpha \\
D &= \frac{1}{\sqrt{2}} g_1 w_{f_{1S}} Z_K Z_{\eta_S} \alpha
\end{aligned}$$

Decay $K_S^E \to K^\star \omega_N$

$$\mathcal{L}_I = A\, K_S^E K_\mu^\star \omega_N^\mu$$

$$A = (\frac{1}{4} h_2^\star + \frac{1}{2} h_3^\star)\phi_N - \frac{1}{4} g_1 g_1^\star \alpha \phi_N + \frac{1}{2\sqrt{2}} h_2^\star \phi_S + \frac{1}{2\sqrt{2}} g_1 g_1^\star \alpha \phi_S$$

Decay $K_S^E \to f_{1N} K$

$$\mathcal{L}_I = A\, K_S^E f_{1N}^\mu (\partial_\mu K) + B\, (\partial_\mu K_S^E) f_{1N}^\mu K$$

$$\begin{aligned}
A &= -\frac{1}{2} g_1^\star Z_K \alpha + (\frac{1}{4} h_2^\star - \frac{1}{2} h_3^\star) w_{K_1} Z_K \phi_N + \frac{3}{4} g_1 g_1^\star w_{K_1} Z_K \alpha \phi_N + \\
&\quad + \frac{1}{2\sqrt{2}} h_2^\star w_{K_1} Z_K \phi_S + \frac{1}{2\sqrt{2}} g_1 g_1^\star w_{K_1} Z_K \alpha \phi_S
\end{aligned}$$

$$B = \frac{1}{2} g_1 Z_K \alpha$$

Decay $\pi^E \to \pi\rho$

$$\mathcal{L}_I = A\, \pi^{0E} (\boldsymbol{\rho}^\mu \partial_\mu \boldsymbol{\pi}) + B\, (\partial_\mu \pi^{0E})(\boldsymbol{\rho}^\mu \boldsymbol{\pi})$$

$$\begin{aligned}
A_{\text{uncharged}} &= 0 \\
A_{\text{charged}} &= \mathrm{i} g_1^\star Z_\pi \alpha + \mathrm{i} h_3^\star w_{a_1} Z_\pi \phi_N - \mathrm{i} g_1 g_1^\star w_{a_1} Z_\pi \alpha \phi_N \\
B_{\text{uncharged}} &= 0 \\
B_{\text{charged}} &= -\mathrm{i} g_1 Z_\pi \alpha
\end{aligned}$$

Decay $K_S^E \to K\pi$

$$\begin{aligned}
\mathcal{L}_I = & A\, K_S^{0E} \bar{K}^0 \pi^0 + B\, K_S^{0E} (\partial_\mu \bar{K}^0)(\partial^\mu \pi^0) \\
& + C\, (\partial_\mu K_S^{0E})(\partial^\mu \bar{K}^0)\pi^0 + D\, (\partial_\mu K_S^{0E}) \bar{K}^0 (\partial^\mu \pi^0)
\end{aligned}$$

$$\begin{aligned}
A &= \frac{1}{2\sqrt{2}} Z_K Z_\pi \xi_1 \phi_S \\
B &= \frac{1}{2} g_1^\star (w_{a_1} + w_{K_1}) Z_K Z_\pi \alpha - \frac{1}{4}(h_2^\star - 2h_3^\star + 3 g_1 g_1^\star \alpha) w_{a_1} w_{K_1} Z_K Z_\pi \phi_N \\
&\quad - \frac{1}{2\sqrt{2}} (h_2^\star + g_1 g_1^\star \alpha) w_{a_1} w_{K_1} Z_K Z_\pi \phi_S \\
C &= -\frac{1}{2} g_1 w_{K_1} Z_K Z_\pi \alpha \\
D &= -\frac{1}{2} g_1 w_{a_1} Z_K Z_\pi \alpha
\end{aligned}$$

Decay $K_S^E \to K a_1$

$$\mathcal{L}_I = A\, K_S^{0E} (\partial_\mu \bar{K}^0) a_1^{0\mu} + B\, (\partial_\mu K_S^{0E}) \bar{K}^0 a_1^{0\mu}$$

$$\begin{aligned}
A &= \frac{1}{2} g_1^\star Z_K \alpha - \frac{1}{4}(h_2^\star - 2h_3^\star + 3 g_1 g_1^\star \alpha) w_{K_1} Z_K \phi_N - \frac{1}{2\sqrt{2}} (h_2^\star + g_1 g_1^\star \alpha) w_{K_1} Z_K \phi_S \\
B &= -\frac{1}{2} g_1 Z_K \alpha
\end{aligned}$$

Decay $K_S^E \to K_1 \eta_N$

$$\mathcal{L}_I = A\, K_S^{0E} \bar{K}_1^{0\mu} (\partial_\mu \eta_N) + B\, (\partial_\mu K_S^{0E}) \bar{K}_1^{0\mu} \eta_N$$

$$\begin{aligned}
A &= -\frac{1}{2} g_1^\star Z_{\eta_N} \alpha + \frac{1}{4}(h_2^\star - 2h_3^\star + 3 g_1 g_1^\star \alpha) w_{f_{1N}} Z_{\eta_N} \phi_N + \frac{1}{2\sqrt{2}} (h_2^\star + g_1 g_1^\star \alpha) w_{f_{1N}} Z_{\eta_N} \phi_S \\
B &= \frac{1}{2} g_1 Z_{\eta_N} \alpha
\end{aligned}$$

Decay $K_S^E \to K_1 \pi$

$$\mathcal{L}_I = A\, K_S^{0E} \bar{K}_1^{0\mu} (\partial_\mu \pi^0) + B\, (\partial_\mu K_S^{0E}) \bar{K}_1^{0\mu} \pi^0$$

$$A = \frac{1}{2} g_1^\star Z_\pi \alpha - \frac{1}{4}(h_2^\star - 2h_3^\star + 3g_1 g_1^\star) w_{a_1} Z_\pi \phi_N - \frac{1}{2\sqrt{2}}(h_2^\star + g_1 g_1^\star \alpha) w_{a_1} Z_\pi \phi_S$$
$$B = -\frac{1}{2} g_1 Z_{\eta_N} \alpha$$

Decay $K_S^E \to \rho K^\star$

$$\mathcal{L}_I = A\, K_S^{0E} \bar{K}_\mu^{\star 0} \rho^{0\mu}$$

$$A = -\frac{1}{4}(h_2^\star + 2h_3^\star - g_1 g_1^\star \alpha)\phi_N - \frac{1}{2\sqrt{2}}(h_2^\star + g_1 g_1^\star \alpha)\phi_S$$

Decay $K_S^E \to K \eta_N^E$

$$\mathcal{L}_I = K_S^{0E} \bar{K}^0 \eta_N^E + B\, K_S^{0E} (\partial_\mu \bar{K}^0)(\partial^\mu \eta_N^E) + C\, (\partial_\mu K_S^{0E})(\partial^\mu \bar{K}^0)\eta_N^E + D\, (\partial_\mu K_S^{0E}) \bar{K}^0 (\partial^\mu \eta_N^E)$$

$$A = \frac{1}{4} Z_K \lambda_2^\star \phi_N - \frac{1}{2} Z_K \xi_2 \phi_N - \frac{1}{2\sqrt{2}} Z_K \lambda_2^\star \phi_S$$
$$B = -\frac{1}{2} g_1^\star w_{K_1} Z_K$$
$$C = \frac{1}{2} g_1^\star w_{K_1} Z_K$$
$$D = 0$$

Decay $K_S^E \to K \pi^E$

$$\mathcal{L}_I = K_S^{0E} \bar{K}^0 \pi^{0E} + B\, K_S^{0E} (\partial_\mu \bar{K}^0)(\partial^\mu \pi^{0E}) + C\, (\partial_\mu K_S^{0E})(\partial^\mu \bar{K}^0)\pi^{0E} + D\, (\partial_\mu K_S^{0E}) \bar{K}^0 (\partial^\mu \pi^{0E})$$

$$A = -\frac{1}{4} Z_K \lambda_2^\star \phi_N + \frac{1}{2} Z_K \xi_2 \phi_N + \frac{1}{2\sqrt{2}} Z_K \lambda_2^\star \phi_S$$
$$B = \frac{1}{2} g_1^\star w_{K_1} Z_K$$
$$C = -\frac{1}{2} g_1^\star w_{K_1} Z_K$$
$$D = 0$$

Decay $K_S^E \to K^E \pi$

$$\mathcal{L}_I = K_S^{0E} \bar{K}^{0E} \pi^0 + B\, K_S^{0E} (\partial_\mu \bar{K}^{0E})(\partial^\mu \pi^0) + C\, (\partial_\mu K_S^{0E})(\partial^\mu \bar{K}^{0E})\pi^0 + D\, (\partial_\mu K_S^{0E}) \bar{K}^{0E} (\partial^\mu \pi^0)$$

$$A = \frac{1}{\sqrt{2}} Z_\pi \xi_2 \phi_S$$
$$B = \frac{1}{2} g_1^\star w_{a_1} Z_\pi$$
$$C = 0$$
$$D = -\frac{1}{2} g_1^\star w_{a_1} Z_\pi$$

Decay $K^E \to K^\star\pi$

$$\mathcal{L}_I = A\,K^{0E}\bar{K}^{\star 0}_\mu \partial_\mu \pi^0 + B\,(\partial^\mu K^{0E})\bar{K}^{\star 0}_\mu \pi^0$$

$$A = -\frac{\mathrm{i}}{2} g_1^\star Z_\pi \alpha + \frac{\mathrm{i}}{4}(h_2^\star - 2h_3^\star + 3g_1 g_1^\star \alpha) w_{a_1} Z_\pi \phi_N - \frac{\mathrm{i}}{2\sqrt{2}}(h_2^\star + g_1 g_1^\star \alpha) w_{a_1} Z_\pi \phi_S$$
$$B = \frac{\mathrm{i}}{2} g_1 Z_\pi \alpha$$

Decay $K^E \to K\omega_N$

$$\mathcal{L}_I = A\,K^{0E}(\partial_\mu \bar{K}^0)\omega_N^\mu + B\,(\partial_\mu K^{0E})\bar{K}^0 \omega_N^\mu$$

$$A = -\frac{\mathrm{i}}{2} g_1^\star Z_K \alpha - \frac{\mathrm{i}}{4}(h_2^\star + 2h_3^\star - g_1 g_1^\star \alpha) w_{K_1} Z_K \phi_N + \frac{\mathrm{i}}{2\sqrt{2}}(h_2^\star + g_1 g_1^\star \alpha) w_{K_1} Z_K \phi_S$$
$$B = \frac{\mathrm{i}}{2} g_1 Z_K \alpha$$

Decay $K^E \to K\rho$

$$\mathcal{L}_I = A\,K^{0E}(\partial_\mu \bar{K}^0)\rho^{0\mu} + B\,(\partial_\mu K^{0E})\bar{K}^0 \rho^{0\mu}$$

$$A = \frac{\mathrm{i}}{2} g_1^\star Z_K \alpha + \frac{\mathrm{i}}{4}(h_2^\star + 2h_3^\star - g_1 g_1^\star \alpha) w_{K_1} Z_K \phi_N - \frac{\mathrm{i}}{2\sqrt{2}}(h_2^\star + g_1 g_1^\star \alpha) w_{K_1} Z_K \phi_S$$
$$B = -\frac{\mathrm{i}}{2} g_1 Z_K \alpha$$

Decay $\sigma_S^E \to \pi\pi^E$

$$\mathcal{L}_I = A\,\sigma_S^E \boldsymbol{\pi}\boldsymbol{\pi}^E + B\,(\partial_\mu \sigma_S^E)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi}^E + B\,(\partial_\mu \sigma_S^E)\boldsymbol{\pi}(\partial^\mu \boldsymbol{\pi}^E) + D\,\sigma_S^E(\partial_\mu \boldsymbol{\pi})(\partial^\mu \boldsymbol{\pi}^E)$$

$$A = -2Z_\pi \kappa_2 \phi_S$$
$$B = C = D = 0$$

Decay $\sigma_N^E \to \sigma_N\pi\pi$

$$\mathcal{L}_I = A\,\sigma_N^E \sigma_N \boldsymbol{\pi}\boldsymbol{\pi} + B\,(\partial_\mu \sigma_N^E)(\partial^\mu \sigma_N)\boldsymbol{\pi}\boldsymbol{\pi} + C\,(\partial_\mu \sigma_N^E)\sigma_N(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi} + \\ + D\,\sigma_N^E(\partial_\mu \sigma_N)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi} + E\,\sigma_N^E \sigma_N (\partial_\mu \boldsymbol{\pi})^2$$

$$A = -\frac{1}{2} Z_\pi^2 \kappa_1 - \frac{1}{4} Z_\pi^2 \xi_1$$
$$B = C = D = 0$$
$$E = \frac{1}{2} w_{a_1}^2 Z_\pi^2 (h_1^\star + h_2^\star - h_3^\star) + g_1 g_1^\star w_{a_1}^2 Z_\pi^2 \alpha$$

Decay $\sigma_S^E \to \sigma_S \pi\pi$

$$\mathcal{L}_I = A\,\sigma_N^E \sigma_N \boldsymbol{\pi}\boldsymbol{\pi} + B\,(\partial_\mu \sigma_N^E)(\partial^\mu \sigma_N)\boldsymbol{\pi}\boldsymbol{\pi} + C\,(\partial_\mu \sigma_N^E)\sigma_N(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi} + \\ + D\,\sigma_N^E(\partial_\mu \sigma_N)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi} + E\,\sigma_N^E \sigma_N (\partial_\mu \boldsymbol{\pi})^2$$

$$A = -\frac{1}{2} Z_\pi^2 \kappa_1$$
$$B = C = D = 0$$
$$E = \frac{1}{2} h_1^\star w_{a_1}^2 Z_\pi^2$$

Decay $\eta_N^E \to \eta_N \pi\pi$

$$\mathcal{L}_I = A\,\eta_N^E \eta_N \boldsymbol{\pi}\boldsymbol{\pi} + B\,(\partial_\mu \eta_N^E)(\partial^\mu \eta_N)\boldsymbol{\pi}\boldsymbol{\pi} + C\,(\partial_\mu \eta_N^E)\eta_N(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi} + \\ + D\,\eta_N^E(\partial_\mu \eta_N)(\partial^\mu \boldsymbol{\pi})\boldsymbol{\pi} + E\,\eta_N^E \eta_N (\partial_\mu \boldsymbol{\pi})^2$$

$$A = -\frac{1}{2} Z_{\eta_N} Z_\pi^2 \kappa_1 - \frac{3}{4} Z_{\eta_N} Z_\pi^2 \xi_1$$
$$B = C = 0$$
$$D = w_{a_1} w_{f_{1N}} Z_{\eta_N} Z_\pi^2 (h_2^\star - h_3^\star) + 2 g_1 g_1^\star w_{a_1} w_{f_{1N}} Z_{\eta_N} Z_\pi^2 \alpha$$
$$E = \frac{1}{2} w_{a_1}^2 Z_{\eta_N} Z_\pi^2 (h_1^\star + h_2^\star - h_3^\star) + g_1 g_1^\star w_{a_1}^2 Z_{\eta_N} Z_\pi^2 \alpha$$

Decay $\eta_N^E \to \pi K K$

$$\mathcal{L}_I = A\,\eta_N^E \pi^0 K K + A_+\,\eta_N^E \pi^+ K^- K^0 + A_-\,\eta_N^E \pi^- K^+ \bar{K}^0 \\ + B\,(\partial_\mu \eta_N^E)(\partial^\mu \boldsymbol{\pi}) K K + C\,(\partial_\mu \eta_N^E)\boldsymbol{\pi}(\partial^\mu K) K \\ + D \eta_N^E (\partial_\mu \pi^0)(\partial^\mu K) K + D_+\,\eta_N^E(\partial_\mu \pi^+)(\partial^\mu K^-) K^0 + D_-\,\eta_N^E(\partial_\mu \pi^-)(\partial^\mu K^+)\bar{K}^0 \\ + E \eta_N^E \pi^0 (\partial^\mu K)(\partial_\mu K) + E_+\,\eta_N^E \pi^+ (\partial^\mu K^-)(\partial_\mu K^0) + E_-\,\eta_N^E \pi^- (\partial^\mu K^+)(\partial_\mu \bar{K}^0)$$

$$A = \frac{1}{2} Z_K^2 Z_\pi \xi_1$$
$$A_+ = A_- = -\frac{1}{\sqrt{2}} Z_K^2 Z_\pi \xi_1$$
$$B = 0$$
$$C = 0$$
$$D = -\frac{1}{4} w_{a_1} w_{K_1} Z_K^2 Z_\pi (h_2^\star - 2h_3^\star) - \frac{3}{4} g_1 g_1^\star w_{a_1} w_{K_1} Z_K^2 Z_\pi \alpha$$
$$D_+ = D_- = -\sqrt{2}\,D$$
$$E = -\frac{1}{2} h_2^\star w_{K_1}^2 Z_K^2 Z_\pi - \frac{1}{2} g_1 g_1^\star w_{K_1}^2 Z_K^2 Z_\pi \alpha$$
$$E_+ = E_- = -\sqrt{2} E$$

Decay $\eta_S^E \to \eta_S\pi\pi$

$$\mathcal{L}_I = A\,\eta_S^E\eta_S\boldsymbol{\pi\pi} + B\,(\partial_\mu\eta_S^E)(\partial^\mu\eta_S)\boldsymbol{\pi\pi} + C\,(\partial_\mu\eta_S^E)\eta_S(\partial^\mu\boldsymbol{\pi})\boldsymbol{\pi} + \\ + D\,\eta_S^E(\partial_\mu\eta_S)(\partial^\mu\boldsymbol{\pi})\boldsymbol{\pi} + E\,\eta_S^E\eta_S(\partial_\mu\boldsymbol{\pi})^2$$

$$\begin{aligned} A &= -\frac{1}{2}Z_{\eta_S}Z_\pi^2\kappa_1 \\ B &= C = D = 0 \\ E &= \frac{1}{2}h_1^\star w_{a_1}^2 Z_{\eta_S}Z_\pi^2 \end{aligned}$$

Decay $\eta_S^E \to \pi KK$

$$\begin{aligned} \mathcal{L}_I = {} & A\,\eta_S^E\pi^0 KK + A_+\,\eta_S^E\pi^+K^-K^0 + A_-\,\eta_S^E\pi^-K^+\bar{K}^0 \\ & + B\,(\partial_\mu\eta_S^E)(\partial^\mu\boldsymbol{\pi})KK + C\,(\partial_\mu\eta_S^E)\boldsymbol{\pi}(\partial^\mu K)K \\ & + D\eta_S^E(\partial_\mu\pi^0)(\partial^\mu K)K + D_+\,\eta_S^E(\partial_\mu\pi^+)(\partial^\mu K^-)K^0 + D_-\,\eta_S^E(\partial_\mu\pi^-)(\partial^\mu K^+)\bar{K}^0 \\ & + E\eta_S^E\pi^0(\partial^\mu K)(\partial_\mu K) + E_+\,\eta_S^E\pi^+(\partial^\mu K^-)(\partial_\mu K^0) + E_-\,\eta_S^E\pi^-(\partial^\mu K^+)(\partial_\mu\bar{K}^0) \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2\sqrt{2}}Z_K^2 Z_\pi\xi_1 \\ A_+ &= A_- = -\sqrt{2}A \\ B &= 0 \\ C &= 0 \\ D &= -\frac{1}{2\sqrt{2}}h_2^\star w_{a_1}w_{K_1}Z_K^2 Z_\pi - \frac{1}{2\sqrt{2}}g_1 g_1^\star w_{a_1}w_{K_1}Z_K^2 Z_\pi\alpha = -D_{\text{charged}} \\ D_+ &= D_- = -\sqrt{2}\,D \\ E &= \frac{1}{\sqrt{2}}h_3^\star w_{K_1}^2 Z_K^2 Z_\pi - \frac{1}{\sqrt{2}}g_1 g_1^\star w_{K_1}^2 Z_K^2 Z_\pi\alpha \\ E_+ &= E_- = -\sqrt{2}E \end{aligned}$$

Decay $a_0^E \to a_0\pi\pi$

$$\mathcal{L}_I = A\,(\mathbf{a}_0^E\mathbf{a}_0)(\boldsymbol{\pi\pi}) + B\,(\mathbf{a}_0^E\boldsymbol{\pi})(\mathbf{a}_0\boldsymbol{\pi}) \\ + C\,(\mathbf{a}_0^E\mathbf{a}_0)\big[(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi})\big] + D\,(\mathbf{a}_0^E\partial_\mu\boldsymbol{\pi})(\mathbf{a}_0\partial^\mu\boldsymbol{\pi})$$

$$\begin{aligned} A &= -\frac{1}{2}\kappa_1 Z_\pi^2 - \frac{3}{4}\xi_1 Z_\pi^2 \\ B &= \frac{1}{2}\xi_1 Z_\pi^2 \\ C &= \frac{1}{2}h_1^\star Z_\pi^2 w_{a_1}^2 + \frac{1}{2}h_2^\star Z_\pi^2 w_{a_1}^2 + \frac{1}{2}h_3^\star Z_\pi^2 w_{a_1}^2 \\ D &= -h_3^\star Z_\pi^2 w_{a_1}^2 + Z_\pi^2 w_{a_1}^2\alpha g_1 g_1^\star \end{aligned}$$

Decay $\pi^E \to \pi\pi\pi$

$$\mathcal{L}_I = A\,(\boldsymbol{\pi}^E\boldsymbol{\pi})(\boldsymbol{\pi}\boldsymbol{\pi}) + B\,(\boldsymbol{\pi}^E\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}) + C\,(\boldsymbol{\pi}^E\boldsymbol{\pi})\Big[(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi})\Big]$$

$$\begin{aligned}
A &= -\frac{1}{2}\kappa_1 Z_\pi^3 - \frac{1}{4}\xi_1 Z_\pi^3 \\
B &= \alpha g_1 g_1^\star Z_\pi^3 w_{a_1}^2 - h_3^\star Z_\pi^3 w_{a_1}^2 \\
C &= \frac{1}{2}h_1^\star Z_\pi^3 w_{a_1}^2 + \frac{1}{2}h_2^\star Z_\pi^3 w_{a_1}^2 + \frac{1}{2}h_3^\star Z_\pi^3 w_{a_1}^2
\end{aligned}$$

Decay $\pi^E \to \pi K K$

$$\begin{aligned}
\mathcal{L}_I = {} & A\,\pi^{0E}\pi^0 KK + A_+\,\pi^{0E}\pi^+K^-K^0 + A_-\,\pi^{0E}\pi^-K^+\bar{K}^0 \\
& + B\,(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi})KK + C\,(\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}(\partial^\mu K)K \\
& + D\pi^{0E}(\partial_\mu\pi^0)(\partial^\mu K)K + D_+\,\pi^{0E}(\partial_\mu\pi^+)(\partial^\mu K^-)K^0 + D_-\,\pi^{0E}(\partial_\mu\pi^-)(\partial^\mu K^+)\bar{K}^0 \\
& + E\pi^{0E}\pi^0(\partial^\mu K)(\partial_\mu K) + E_+\,\pi^{0E}\pi^+(\partial^\mu K^-)(\partial_\mu K^0) + E_-\,\pi^{0E}\pi^-(\partial^\mu K^+)(\partial_\mu\bar{K}^0)
\end{aligned}$$

$$\begin{aligned}
A &= -Z_K^2 Z_\pi \kappa_1 - \frac{1}{2}Z_K^2 Z_\pi \xi_1 \\
A_+ &= A_- = 0 \\
B &= 0 \\
C &= 0 \\
D &= \frac{1}{4}w_{a_1}w_{K_1}Z_K^2 Z_\pi(h_2^\star - 2h_3^\star + 3g_1 g_1^\star\alpha) \\
D_+ &= D_- = -\frac{1}{2\sqrt{2}}w_{a_1}w_{K_1}Z_K^2 Z_\pi(h_2^\star + 2h_3^\star - g_1 g_1^\star\alpha) \\
E &= w_{K_1}^2 Z_K^2 Z_\pi(h_1^\star + \frac{1}{2}h_2^\star) + \frac{1}{2}g_1 g_1^\star w_{K_1}^2 Z_K^2 Z_\pi\alpha \\
E_+ &= E_- = 0
\end{aligned}$$

Decay $K_S^E \to K_S\pi\pi$

$$\begin{aligned}
\mathcal{L}_I = {} & A_1\,K_S^{0E}\bar{K}_S^0\pi^0\pi^0 + A_2\,K_S^{0E}\bar{K}_S^0\pi^+\pi^- + A_3\,K_S^{0E}K_S^-\pi^+\pi^0 \\
& + B_1\,K_S^{0E}(\partial_\mu\bar{K}_S^0)(\partial^\mu\pi^0)\pi^0 + B_2\,K_S^{0E}(\partial_\mu\bar{K}_S^0)(\partial^\mu\pi^+)\pi^- + B_3\,K_S^{0E}(\partial_\mu\bar{K}_S^0)(\partial^\mu\pi^-)\pi^+ \\
& + B_4\,K_S^{0E}(\partial_\mu K_S^-)(\partial^\mu\pi^+)\pi^0 + B_5\,K_S^{0E}(\partial_\mu K_S^-)(\partial^\mu\pi^0)\pi^+ \\
& + C_1\,K_S^{0E}\bar{K}_S^0(\partial_\mu\pi^0)(\partial^\mu\pi^0) + C_2\,K_S^{0E}\bar{K}_S^0(\partial_\mu\pi^+)(\partial_\mu\pi^-) + C_3\,K_S^{0E}K_S^-(\partial_\mu\pi^+)(\partial^\mu\pi^0)
\end{aligned}$$

$$\begin{aligned}
A_1 &= -\frac{1}{2}Z_{K_S}Z_\pi^2\kappa_1 - \frac{1}{4}Z_{K_S}Z_\pi^2\xi_1 \\
A_2 &= 2A_1 \\
A_3 &= 0 \\
B_1 &= \frac{\mathrm{i}}{4}w_{a_1}w_{K^*}^* Z_{K_S}Z_\pi^2(h_2^\star - 2h_3^\star + 3g_1 g_1^\star\alpha)
\end{aligned}$$

$$
\begin{aligned}
B_2 &= \mathrm{i} w_{a_1} w_{K^*}^* Z_{K_S} Z_\pi^2 (-h_3^\star + g_1 g_1^\star \alpha) \\
B_3 &= \frac{\mathrm{i}}{2} w_{a_1} w_{K^*}^* Z_{K_S} Z_\pi^2 (h_2^\star + g_1 g_1^\star \alpha) \\
B_4 &= \frac{\mathrm{i}}{2\sqrt{2}} w_{a_1} w_{K^*}^* Z_{K_S} Z_\pi^2 (-h_2^\star - 2h_3^\star + g_1 g_1^\star \alpha) \\
B_5 &= -B_4 \\
C_1 &= \frac{1}{4} w_{a_1}^2 Z_{K_S} Z_\pi^2 (2h_1^\star + h_2^\star + g_1 g_1^\star \alpha) \\
C_2 &= 2C_1 \\
C_3 &= 0
\end{aligned}
$$

Decay $K^E \to K\pi\pi$

$$
\begin{aligned}
\mathcal{L}_I = {} & A_1\, K^{0E} \bar{K}^0 \pi^0 \pi^0 + A_2\, K^{0E} \bar{K}^0 \pi^+ \pi^- + A_3\, K^{0E} K^- \pi^+ \pi^0 \\
& + B_1\, K^{0E} (\partial_\mu \bar{K}^0)(\partial^\mu \pi^0)\pi^0 + B_2\, K^{0E}(\partial_\mu \bar{K}^0)(\partial^\mu \pi^+)\pi^- + B_3\, K^{0E}(\partial_\mu \bar{K}^0)(\partial^\mu \pi^-)\pi^+ \\
& + B_4\, K^{0E}(\partial_\mu K^-)(\partial^\mu \pi^+)\pi^0 + B_5\, K^{0E}(\partial_\mu K^-)(\partial^\mu \pi^0)\pi^+ \\
& + C_1\, K^{0E} \bar{K}^0 (\partial_\mu \pi^0)(\partial^\mu \pi^0) + C_2\, K^{0E} \bar{K}^0 (\partial_\mu \pi^+)(\partial_\mu \pi^-) + C_3\, K^{0E} K^- (\partial_\mu \pi^+)(\partial^\mu \pi^0)
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= -\frac{1}{2} Z_K Z_\pi^2 \kappa_1 - \frac{1}{4} Z_K Z_\pi^2 \xi_1 \\
A_2 &= 2A_1 \\
A_3 &= 0 \\
B_1 &= \frac{1}{4} w_{a_1} w_{K_1} Z_K Z_\pi^2 (h_2^\star - 2h_3^\star + 3g_1 g_1^\star \alpha) \\
B_2 &= w_{a_1} w_{K_1} Z_K Z_\pi^2 (-h_3^\star + g_1 g_1^\star \alpha) \\
B_3 &= \frac{1}{2} w_{a_1} w_{K_1} Z_K Z_\pi^2 (h_2^\star + g_1 g_1^\star \alpha) \\
B_4 &= \frac{1}{2\sqrt{2}} w_{a_1} w_{K_1} Z_K Z_\pi^2 (-h_2^\star - 2h_3^\star + g_1 g_1^\star \alpha) \\
B_5 &= -B_4 \\
C_1 &= \frac{1}{4} w_{a_1}^2 Z_K Z_\pi^2 (2h_1^\star + h_2^\star + g_1 g_1^\star \alpha) \\
C_2 &= 2C_1 \\
C_3 &= 0
\end{aligned}
$$

Decay $K^E \to K\pi\eta_N$

$$
\begin{aligned}
\mathcal{L}_I = {} & A\, K^{0E} \bar{K}^0 \pi^0 \eta_N + A_+\, K^{0E} K^- \pi^+ \eta_N \\
& + B\, K^{0E}(\partial_\mu \bar{K}^0)(\partial^\mu \pi^0)\eta_N + B_+\, K^{0E}(\partial_\mu K^-)(\partial^\mu \pi^+)\eta_N \\
& + C\, K^{0E}(\partial_\mu \bar{K}^0)\pi^0(\partial^\mu \eta_N) + C_+\, K^{0E}(\partial_\mu K^-)\pi^+(\partial^\mu \eta_N) \\
& + D\, K^{0E} \bar{K}^0 (\partial_\mu \pi^0)(\partial^\mu \eta_N) + D_+\, K^{0E} K^- (\partial_\mu \pi^+)(\partial^\mu \eta_N)
\end{aligned}
$$

$$
A = \frac{1}{2} Z_K Z_{\eta_N} Z_\pi \xi_1
$$

$$
\begin{aligned}
A_+ &= -\sqrt{2}A \\
B &= \frac{1}{4} w_{a_1} w_{K_1} Z_K Z_{\eta_N} Z_\pi (-h_2^\star + 2h_3^\star - 3g_1 g_1^\star \alpha) \\
B_+ &= -\sqrt{2}B \\
C &= \frac{1}{4} w_{f_{1N}} w_{K_1} Z_K Z_{\eta_N} Z_\pi (-h_2^\star + 2h_3^\star - 3g_1 g_1^\star \alpha) \\
C_+ &= -\sqrt{2}C \\
D &= -\frac{1}{2} w_{a_1} w_{f_{1N}} Z_K Z_{\eta_N} Z_\pi (h_2^\star + g_1 g_1^\star \alpha) \\
D_+ &= -\sqrt{2}D
\end{aligned}
$$

Bibliography

[1] C. Patrignani and P. D. Group, "Review of Particle Physics", *Chinese Phys. C* **40** (2016), 100001.

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books Publishing, 1995), ISBN:0-201-50397.

[3] D. Parganlija, "Quarkonium Phenomenology in Vacuum", *PhD Thesis* (2012), arXiv:1208.0204.

[4] J. Bai and [BES-Collaboration], "Partial wave analysis of $J/\psi \to \gamma(K^+K^-\pi^0)$", *Phys. Lett. B* **440** (1998), 217–224, DOI:10.1016/S0370-2693(98)01241-6.

[5] M. Ablikim and [BES-Collaboration], "Resonances in $J/\psi \to \phi\pi^+\pi^-$ and ϕK^+K^-", *Phys. Lett. B* **607** (2005), 243–253, DOI:10.1016/j.physletb.2004.12.041.

[6] J. Bai and [BES-Collaboration], "Partial wave analysis of $J/\psi \to \gamma(K^\pm K^0_S\pi^\mp)$", *Phys. Lett. B* **476** (2000), 25–32, DOI:10.1016/S0370-2693(00)00115-5.

[7] F. Giacosa, "Theoretical Errors", (2013), URL:https://th.physik.uni-frankfurt.de/~giacosa/decays-Dateien/error.pdf.