PROJEKTARBEIT

Low-Energy Dynamics of 2D SUSY Gauge Theories

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I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have acknowledged all the sources of information which have been used in the thesis.

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Chapter 1

Introduction

Supersymmetric gauge theories in various dimensions play an important role in modern theoretical physics. In this work, we investigate the gauged linear sigma model (GLSM) [1], which is a supersymmetric gauge theory in two dimensions. This theory is of particular interest in the context of string theory: the vacuum configurations of GLSMs with particular choices of gauge groups and matter content are Calabi-Yau spaces. These are the most prominent candidates for the "extra dimensions" in compactifications of string theory from ten to four dimensions. Calabi-Yaus come with free parameters called moduli. A certain class of moduli (the Kähler parameters) can be identified with coupling constants (the FI-theta parameters) of the GLSM. The Kähler moduli space has a non-trivial mathematical structure. In particular there are singularities. In the context of the GLSM, these are determined by the effective potential $\widetilde{W}_{\text{eff}}$ on the Coulomb and mixed branches of the theory. The main goal of this work is to derive $\widetilde{W}_{\text{eff}}$ on the Coulomb branch for the case of a GLSM with gauge group $U(1)$ [2].

In Chapter 2, we introduce our notation and present the most common aspects of a supersymmetric gauge theory. Chapter 3 contains the explicit calculations to arrive at the quantum effective potential. In the chapters above, we summarize the main steps. The explicit calculations can be found in the appendix.

Chapter 2

Supersymmetry

In this chapter, we introduce our notation for supersymmetric theories. Section 2.1 defines the superspace and superfields we will work with. Section 2.3 introduces supersymmetric gauge field theories and Section 2.2 describes how to build a Lagrangian density that is invariant under SUSY transformations. This chapter is mainly based on [2].

2.1 $\mathcal{N} = (2, 2)$ Superspace

The superspace formalism extends a d-dimensional Minkowski space with bosonic coordinates x^μ to a superspace by also including fermionic coordinates θ. In particular, for an $\mathcal{N} = (2, 2)$ supersymmetric theory, we define two fermionic coordinates θ^a and their conjugates $\overline{\theta}^{\dot{a}}$ $(a, \dot{a} = \pm)$. They are related via complex conjugation $(\theta^\pm)^* = \overline{\theta}^\pm$ with $(\theta\theta')^* = \overline{\theta}'\overline{\theta}$. A supersymmetric transformation can be described by a translation in superspace. The generators of such a transformation are given by:

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm, \quad \overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - \mathrm{i}\theta^\pm\partial_\pm, \tag{2.1}$$

where $\partial_\pm$ are derivatives with respect to the coordinates $x^\pm \equiv x^0 \pm x^1$:

$$\partial_\pm \equiv \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{2.2}$$

The fermionic derivatives are not supersymmetrically covariant. This means, derivatives of a superfield with respect to the fermionic coordinates are not themselves superfields: $\delta_\epsilon\left(\frac{\partial *}{\partial\theta^\pm}\right) \neq \frac{\partial}{\partial\theta^\pm}(\delta_\epsilon *)$. To remedy this, we define covariant derivatives,

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm, \quad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\partial_\pm, \tag{2.3}$$

which obey $\delta_\epsilon\left(D_\pm *\right) = D_\pm(\delta_\epsilon *)$, since all anticommutators of a SUSY generator $\mathcal{Q}$ with a covariant derivative D vanish:

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} \overset{A.1}{=} -2\mathrm{i}\partial_\pm, \quad \{D_\pm, \overline{D}_\pm\} = 2\mathrm{i}\partial_\pm, \quad \{D_\pm, \mathcal{Q}_\pm\} = 0. \tag{2.4}$$

Because of their Grassmannian nature ($\theta^\pm\theta^\pm = 0$, $\overline{\theta}^\pm\overline{\theta}^\pm = 0$), every function in our superspace – called a superfield – can be represented by a finite Taylor expansion in θ and $\overline{\theta}$. For a general superfield $\mathcal{F} = \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm)$, such an expansion reads,

$$\mathcal{F}(x, \theta, \overline{\theta}) = f(x) + \theta^+\psi_+(x) + \theta^-\psi_-(x) + \overline{\theta}^+\overline{\chi}_+(x) + \overline{\theta}^-\overline{\chi}_-(x) + \theta^+\theta^-g_{+-}(x) + \ldots$$

Here, bosonic and fermionic fields together build up the superfield. A SUSY transformation acts on a superfield in the following way,

$$\mathcal{F} \to \mathcal{F}' = \mathcal{F} + \delta_\epsilon \mathcal{F}, \qquad \text{with } \delta_\epsilon = \epsilon^a Q_a + \bar{\epsilon}^{\dot{a}} \overline{Q}_{\dot{a}} \\ = \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \bar{\epsilon}_+ \overline{\mathcal{Q}}_- + \bar{\epsilon}_- \overline{\mathcal{Q}}_+ \tag{2.5}$$

Since the generators are spinors themselves, the infintesimal parameter for SUSY transformations, $\epsilon^a = (\epsilon^-, \epsilon^+)^\mathsf{T}$ and its conjugate, must be spinors as well.
In addition to this most general superfield $\mathcal{F}$, we can define further, more restricted superfields.

Chiral superfield. We can constrain superfields in several ways. If we demand,

$$\overline{D}_\pm \Phi = 0, \tag{2.6}$$

then Φ is called a chiral superfield. It is convenient to define $y^\pm := x^\pm - \mathrm{i}\theta^\pm\bar{\theta}^\pm$ for which

$$\overline{D}_\pm (x^\pm - \mathrm{i}\theta^\pm\bar{\theta}^\pm) = 0 \quad \text{and} \quad \overline{D}_\pm \theta^\pm = 0. \tag{2.7}$$

Using these coordinates, the chiral superfield reads:

$$\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) \overset{A.2}{=} \phi(y^\pm) + \theta^a \psi_a(y^\pm) + \theta^+\theta^- F(y^\pm), \tag{2.8}$$

and to write it in terms of the usual coordinates $(x, \theta, \bar{\theta})$, we have to perform a Taylor expansion:

$$\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) \overset{A.3}{=} \phi(x^\pm) - \mathrm{i}\theta^+\bar{\theta}^+ \partial_+\phi(x^\pm) - \mathrm{i}\theta^-\bar{\theta}^- \partial_-\phi(x^\pm) - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+ \partial_+\partial_-\phi(x^\pm) \\ + \theta^+\psi_+(x^\pm) - \mathrm{i}\theta^+\theta^-\bar{\theta}^- \partial_-\psi_+(x^\pm) + \theta^-\psi_-(x^\pm) \\ - \mathrm{i}\theta^-\theta^+\bar{\theta}^+ \partial_+\psi_-(x^\pm) + \theta^+\theta^- F(x^\pm). \tag{2.9}$$

The degrees of freedom in the left chiral superfield are a complex scalar field ϕ, a spinor field $\psi = (\psi_-, \psi_+)^\mathsf{T}$ and an auxiliary field F. Similarly, one can define an anti-chiral superfield via $D_\pm \overline{\Phi} = 0$. It is given by complex conjugation:

$$\overline{\Phi}(x^\mu, \theta^\pm, \bar{\theta}^\pm) = \bar{\phi}(x^\pm) + \mathrm{i}\theta^+\bar{\theta}^+ \partial_+\bar{\phi}(x^\pm) + \mathrm{i}\theta^-\bar{\theta}^- \partial_-\bar{\phi}(x^\pm) - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+ \partial_+\partial_-\bar{\phi}(x^\pm) \\ - \bar{\theta}^+\bar{\psi}_+(x^\pm) - \mathrm{i}\bar{\theta}^+\theta^-\bar{\theta}^- \partial_-\bar{\psi}_+(x^\pm) - \bar{\theta}^-\bar{\psi}_-(x^\pm) \\ - \mathrm{i}\bar{\theta}^-\theta^+\bar{\theta}^+ \partial_+\bar{\psi}_-(x^\pm) + \bar{\theta}^-\bar{\theta}^+ \overline{F}(x^\pm). \tag{2.10}$$

Twisted chiral superfields. Mirror symmetry transforms chiral multiplets into twisted chiral multiplets U. They are defined via $\overline{D}_+ U = D_- U = 0$. Again, similarly one can define a twisted anti-chiral superfield $\overline{U}$ via $\overline{D}_- \overline{U} = D_+ \overline{U} = 0$. Twisted chiral superfields will play an important role in the next chapter, since they enable us to use additional terms in a Lagrangian.

Real superfields. Real (or vector) superfields V are constrained by $V = V^*$. They will play an important role when we consider gauge field theories in a later section. It is useful to perform a certain gauge transformation and represent the vector field in the so-called Wess-Zumino gauge:

$$V_{\mathrm{WZ}} \overset{(A.4)}{=} \theta^-\bar{\theta}^-(v_0 - v_1) + \theta^+\bar{\theta}^+(v_0 + v_1) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} \\ + \mathrm{i}\theta^-\theta^+\bar{\theta}^a\bar{\lambda}_a + \mathrm{i}\bar{\theta}^+\bar{\theta}^-\theta^a\lambda_a + \theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D \tag{2.11}$$

In this gauge, one can easily show that $V^n = 0$ for $n > 2$.

2.2 Supersymmetric Lagrangians

In order to build a supersymmetric Lagrangian, it is convenient to notice that an integral over bosonic and fermionic coordinates,

$$\int \mathrm{d}^2x \int \mathrm{d}^4\theta\, \mathcal{F}(\mathcal{S}), \quad \mathrm{d}^4\theta = \mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\overline{\theta}^-\,\mathrm{d}\overline{\theta}^+ \tag{2.12}$$

is supersymmetrically invariant for any function $\mathcal{F}$ of superfields $\mathcal{S}$ (Proof: Appendix A.5). This is because $\delta_\epsilon\mathcal{S}$ which contains the SUSY generators, is built out of total derivatives. Equation (2.12) describes an action, so for the Lagrangian, we only consider integration over θ and $\overline{\theta}$. We define a D-term

$$\int \mathrm{d}^4\theta\, \mathcal{S} \equiv [\mathcal{S}]_D, \tag{2.13}$$

which yields the coefficient of $\theta^+\theta^-\overline{\theta}^-\overline{\theta}^+$ in a Taylor expansion of $\mathcal{S}$. Next we define an F-term,

$$\int \mathrm{d}^2\theta\, \Phi\Big|_{\overline{\theta}^\pm=0} \equiv [\Phi]_F, \quad \mathrm{d}^2\theta = \mathrm{d}\theta^-\mathrm{d}\theta^+, \tag{2.14}$$

based on the chiral superfield Φ. Similarly to the regular F-term, we define a twisted F-term using a twisted chiral superfield U,

$$\int \widetilde{\mathrm{d}^2\theta} U \equiv [U]_{\widetilde{F}}, \quad \widetilde{\mathrm{d}^2\theta} = \mathrm{d}\overline{\theta}^-\mathrm{d}\theta^+. \tag{2.15}$$

Since a function of a (twisted) chiral superfield is again a (twisted) chiral superfield, we can write a more general Lagrangian,

$$\mathcal{L} = [\mathcal{F}(\mathcal{S})]_D + \left[W(\Phi) + c.c.\right]_F + \left[\widetilde{W}(U) + c.c.\right]_{\widetilde{F}}, \tag{2.16}$$

where $\mathcal{F}$ is an arbitrary, differentiable function and W, $\widetilde{W}$ are holomorphic functions. *c.c.* denotes complex conjugates, which are calculated via $\mathrm{d}^2\overline{\theta} = \mathrm{d}\theta^+\mathrm{d}\theta^-$ and $\widetilde{\mathrm{d}^2\overline{\theta}} = \mathrm{d}\overline{\theta}^+\mathrm{d}\theta^-$.

2.3 Supersymmetric Gauge Theories

One can show that the Lagrangian

$$\mathcal{L} = [\overline{\Phi}\Phi]_D \tag{2.17}$$

leads to the standard kinetic term for a complex scalar field and fermionic fields. It is invariant under a constant $U(1)$ phase shift $\Phi \to \mathrm{e}^{\mathrm{i}\alpha}\Phi$. In order to arrive at a gauge theory, we promote the parameter α to a function on the superspace, i.e. a superfield: $\alpha \to A(x^\mu, \theta^\pm, \overline{\theta}^\pm)$. This leads to the transformation $\overline{\Phi}\Phi \to \overline{\Phi}\mathrm{e}^{-\mathrm{i}\overline{A}+\mathrm{i}A}\Phi$. We choose A to be a chiral superfield, then $A\Phi$ will also be a chiral superfield. In order to cancel this additional factor, we introduce a real superfield $V = V^*$, that transforms under a gauge transformation as

$$V \to V + \mathrm{i}\overline{A} - \mathrm{i}A. \tag{2.18}$$

The new Lagrangian $\mathcal{L} = [\overline{\Phi}e^V\Phi]_D$ is now invariant under gauge transformations as well as supersymmetric transformations. In order to introduce dynamics for the gauge field V, we define a field strength:

$$\Sigma = \overline{D}_+ D_- V. \tag{2.19}$$

This is a twisted chiral superfield, since

$$\overline{D}_+\Sigma \overset{A.6}{=} 0 = D_-\Sigma, \tag{2.20}$$

and invariant under gauge transformations:

$$\Sigma \xrightarrow{\text{gauge}} \Sigma' \overset{A.7}{=} \Sigma \tag{2.21}$$

In components of the gauge field V, the super field strength is given by

$$\Sigma(\tilde{y}^\pm) \overset{A.8}{=} \sigma(\tilde{y}^\pm) + i\theta^+\overline{\lambda}_+(\tilde{y}^\pm) - i\overline{\theta}^-\lambda_-(\tilde{y}^\pm) + \theta^+\overline{\theta}^-\left[D(\tilde{y}^\pm) - iv_{01}(\tilde{y}^\pm)\right], \tag{2.22}$$

with $\tilde{y}^\pm \equiv x^\pm \mp i\theta^\pm\overline{\theta}^\pm$, $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. Since the super field stength is a twisted chiral superfield, it can be included in a Lagrangian via a D-term or a twisted F-Term:

$$\mathcal{L} \sim [\overline{\Sigma}\Sigma]_D + [\Sigma + c.c.]_{\overline{F}}, \tag{2.23}$$

the complex conjugate ensures that the Lagrangian is real.

Chapter 3

Low-Energy Dynamics

In this chapter, we look at the low energy regime of a two-dimensional $\mathcal{N} = (2,2)$ supersymmetric $U(1)$ gauge theory and calculate the quantum effective potential at one-loop order. To do this, we consider the Coulomb branch of the theory and integrate out the heavy matter field degrees of freedom.
Section 3.1 presents the supersymmetric, gauge invariant Lagrangian we use in the following calculations. In Section 3.2 we define an effective action that enables us to calculate the effective super potential. Finally, in Section 3.3, we calculate the effective super potential itself.

3.1 The Lagrangian

For a $U(1)$ theory with one chiral matter multiplet Φ, we can have D-terms and twisted F-terms, but no F-terms. In an F-term, the chiral superfield (or a function thereof) appears, which cannot be gauge invariant by itself[1]. We construct our Lagrangian using a kinetic term, a gauge field term (both of which are D-terms) and a twisted F-term, the so-called FI-ϑ-term:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{FI},\vartheta} \\ &= \left[\overline{\Phi}\mathrm{e}^{V}\Phi\right]_D - \frac{1}{2e^2}\left[\overline{\Sigma}\Sigma\right]_D - \frac{1}{2}\left[t\Sigma + c.c.\right]_{\widetilde{F}}\end{aligned}$$

Here, e is a coupling constant of one mass dimension and $t = r - \mathrm{i}\vartheta$ is a dimensionless, complex parameter. r is called the Fayet-Iliopoulos parameter and ϑ is called the theta angle.

Kinetic Term. In order to write down the kinetic term, we need to evaluate the exponential function of the super gauge field V. In Wess-Zumino gauge, this is easily done:

$$V^2 \overset{A.9}{=} 2\theta^-\theta^+\overline{\theta}^+\overline{\theta}^-(v^\mu v_\mu - |\sigma|^2), \qquad V^3 = 0 \tag{3.1}$$

Thus, the kinetic term in the Lagrangian is given by the following D-term:

$$\left[\overline{\Phi}(1 + V + \frac{1}{2}V^2)\Phi\right]_D \tag{3.2}$$

[1] Note that if we inlcude more than one chiral matter multiplet, we can build a potential term in a way that the gauge charges of the different multiplets cancel each other, which results in gauge invariance; see [1], Eq. (2.20).

which yields

$$\begin{aligned}\mathcal{L}_{\text{kin}} \overset{A.10}{=} & D^*_\mu\overline{\phi}D^\mu\phi - \mathrm{i}\overline{\psi}_-(D_0+D_1)\psi_- - \mathrm{i}\overline{\psi}_+(D_0-D_1)\psi_+ \\ & + D|\phi|^2 + |F|^2 - |\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- \\ & - \mathrm{i}\overline{\phi}\lambda_-\psi_+ + \mathrm{i}\overline{\phi}\lambda_+\psi_- + \mathrm{i}\overline{\psi}_+\overline{\lambda}_-\phi - \mathrm{i}\overline{\psi}_-\overline{\lambda}_+\phi \end{aligned} \tag{3.3}$$

The covariant derivative is given by $D_\mu = \partial_\mu + \mathrm{i}v_\mu$. For completeness, the gauge- and FI-ϑ-term in the Lagrangian are given by [2]

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= \frac{1}{2e^2}\big[-\partial_\mu\overline{\sigma}\partial^\mu\sigma + \mathrm{i}\overline{\lambda}_-(\partial_0+\partial_1)\lambda_- + \mathrm{i}\overline{\lambda}_+(\partial_0-\partial_1)\lambda_+ + v_{01}^2 + D^2\big], \\ \mathcal{L}_{\text{FI},\vartheta} &= -rD + \vartheta v_{01}.\end{aligned}$$

3.2 Effective Action

From here on we investigate what happens for a large super field strength Σ. The lowest component of Σ is σ, which can be seen as the mass for the component fields of the matter multiplet Φ. For large Σ, the components of Φ are heavy and can thus be integrated out. This is called the Coulomb branch of the theory.
The effective action for the super field strength Σ is therefore defined via,

$$\mathrm{e}^{\mathrm{i}S_{\text{eff}}[\Sigma]} := \int \mathcal{D}\Phi\, \mathrm{e}^{\mathrm{i}S[\Phi,\Sigma]}, \tag{3.4}$$

where Φ is the matter-field multiplet, containing a scalar field ϕ, fermionic fields ψ and an auxiliary field F. The path integral in Equation (3.4) is equivalent to $\exp(\mathrm{i}\sum\mathcal{G}_j)$ where $\mathcal{G}_j$ are connected Feynman diagrams. Since the effective action should be still supersymmetric and gauge invariant, we can make an ansatz regarding its shape. Our degrees of freedom are the fields contained in Σ, this means we can build a D-term and a twisted F-term like this:

$$\mathcal{L}_{\text{eff}}[\Sigma] = \left[K_{\text{eff}}(\Sigma,\overline{\Sigma})\right]_D - \frac{1}{2}\left[\widetilde{W}_{\text{eff}}(\Sigma) + c.c.\right]_{\widetilde{F}} \tag{3.5}$$

where K_{eff} is the effective Kähler potential and $\widetilde{W}_{\text{eff}}$ is the effective super potential[2]. In order to calculate these two functions, we perform two steps: First, we determine how K_{eff} and $\widetilde{W}_{\text{eff}}$ are proportional to D, the auxiliary field appearing in Σ. Then we explicitly calculate the effective action up to one-loop order and identify certain coefficients of this field D with those two functions. Performing the explicit θ-integration in Equation (3.5) leads to

$$[K_{\text{eff}}[\Sigma]]_D \overset{A.11}{\propto} \partial_\sigma\partial_{\overline{\sigma}}K_{\text{eff}}[\sigma]\,|D-\mathrm{i}v_{01}|^2 + \ldots, \tag{3.6}$$

$$\frac{1}{2}[\widetilde{W}_{\text{eff}}[\Sigma] + c.c.]_{\widetilde{F}} \overset{A.12}{\propto} D\,\mathfrak{Re}\big\{\partial_\sigma\widetilde{W}_{\text{eff}}[\sigma]\big\} + v_{01}\mathfrak{Im}\big\{\partial_\sigma\widetilde{W}_{\text{eff}}[\sigma]\big\} + \ldots \tag{3.7}$$

This means that the effective Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_{\text{eff}}[\Sigma] = {} & \partial_\sigma\partial_{\overline{\sigma}}K_{\text{eff}}[\sigma]\,|D-\mathrm{i}v_{01}|^2 \\ & - D\,\mathfrak{Re}\big\{\partial_\sigma\widetilde{W}_{\text{eff}}[\sigma]\big\} - v_{01}\mathfrak{Im}\big\{\partial_\sigma\widetilde{W}_{\text{eff}}[\sigma]\big\} + \ldots\end{aligned} \tag{3.8}$$

So if we calculate the action in terms of the path integral, we can identify those terms that are linear or quadratic in D and v_{01}, which enables us to calculate the two effective potentials.

[2] We chose different signs as compared to Ref. [2], since our metric is mostly-minus and theirs is mostly plus.

Axial transformation. Before doing the calculations, we want to express the complex scalar field σ through its absolute value. We take $\sigma = |\sigma|e^{i\varphi_\sigma}$ and consider terms that are affected by this phase. In the kinetic Lagrangian, σ appears interacting with the spinor field ψ:

$$\mathcal{L}_{\text{kin}} = \ldots - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- \ldots \tag{3.9}$$

In order to write this in terms of $|\sigma|$, we can absorb the phase φ_σ into the spinor fields:

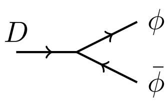
$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \ldots - \overline{\psi}_-|\sigma|e^{i\varphi_\sigma}\psi_+ - \overline{\psi}_+|\sigma|e^{-i\varphi_\sigma}\psi_- \ldots & (3.10)\\ &= \ldots - \overline{\psi}_-e^{i\varphi_\sigma/2}|\sigma|e^{i\varphi_\sigma/2}\psi_+ - \overline{\psi}_+e^{-i\varphi_\sigma/2}|\sigma|e^{-i\varphi_\sigma/2}\psi_- \ldots & (3.11)\end{aligned}$$

This can be seen as an axial $U(1)$ transformation by a phase $\alpha = -\frac{\varphi_\sigma}{2}$. The resulting current is only classically conserved, that is we encounter an anomaly $\partial_\mu j^\mu_A \neq 0$, which accounts for an additional term in the Lagrangian:

$$\mathcal{L} \xrightarrow{\text{axial}} \mathcal{L}' \overset{A.13}{=} \mathcal{L} + \frac{\varphi_\sigma}{2\pi}v_{01}. \tag{3.12}$$

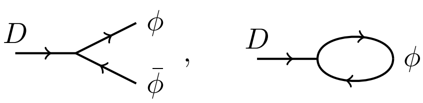
This is equivalent to claiming that the FI-theta angle gets shifted, $\vartheta \to \vartheta + \frac{\varphi_\sigma}{2\pi}$, since there is the term ϑv_{01} in the FI-theta part of the Lagrangian. For a detailed calculation, see Appendix A.13.

Terms linear in D. From the previous section we know that in order to calculate the effective super potential $\widetilde{W}_{\text{eff}}$ and the effective Kähler potential K_{eff}, we need the terms in the effective action linear and quadric in D, respectively. Looking at the Lagrangian (3.3), we see that the field D interacts only with the scalar matter field ϕ. Therefore, the only vertex we need is,



(3.13)

with vertex factor v.f. $=$ i. The possible graphs containing one D-field up to one-loop order are listed below,



(3.14)

but since we want to integrate over $\Phi = \{\phi, \psi, F\}$, there cannot be any external fields ϕ. Thus, the only graph linear in D is the second one. The σ field acts as mass for the complex scalar field ϕ. The propagator for a complex scalar field with mass $m = \sigma$ reads

$$G_\phi(p) = \frac{1}{p^2 - |\sigma|^2 + i\epsilon}. \tag{3.15}$$

The integral representation for the above diagram is given by:

$$S_{\text{eff}}[\Sigma]\big|_D = \int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2 - |\sigma|^2 + i\epsilon}. \tag{3.16}$$

Here, $k^2 = k_\mu k^\mu = (k^0)^2 - (k^1)^2$. We perform a Wick rotation to a Euclidean metric[3] such that $k^0 \to \mathrm{i}k^0 =: k^2$ and $\mathrm{d}k^0 \to -\mathrm{i}\mathrm{d}k^2$. Now, $k^2 = k_\mu k^\mu = (\mathrm{i}k^2)^2 - (k^1)^2 = -(k^1)^2 - (k^2)^2$ and we can substitute polar coordinates $(k^1, k^2) \to (k, \varphi)$. The integral now reads (we dropped the ϵ term, since we will not be encountering any poles from here on),

$$S_{\text{eff}}[\Sigma]\big|_D = -\int \frac{k\,\mathrm{d}k\mathrm{d}\varphi}{(2\pi)^2}\frac{1}{k^2+|\sigma|^2}. \tag{3.17}$$

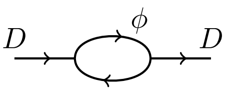
The angular integration evaluates to 2π and for the radial momentum integration, we substitute $u = k^2 + |\sigma|^2$, s.t. $\mathrm{d}u = 2k\mathrm{d}k$,

$$S_{\text{eff}}[\Sigma]\big|_D = -\int_{u_1}^{u_2} \frac{\mathrm{d}u}{4\pi}\frac{1}{u}. \tag{3.18}$$

We include an upper cut-off scale Λ for the momentum integration, which yields the final result,

$$S_{\text{eff}}[\Sigma]\big|_D = -\frac{1}{4\pi}\log\left(\frac{u_2}{u_1}\right) = -\frac{1}{4\pi}\log\left(\frac{\Lambda^2+|\sigma|^2}{|\sigma|^2}\right). \tag{3.19}$$

Terms quadratic in D. The D-quadratic terms are calculated in a similar way: The only one-loop diagram containing two D-fields is the following,



, (3.20)

and the corresponding integral representation reads,

$$S_{\text{eff}}[\Sigma]\big|_{D^2} = \frac{1}{2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{i}^2}{(k^2-|\sigma|^2+\mathrm{i}\epsilon)(p^2-|\sigma|^2+\mathrm{i}\epsilon)}\delta^{(2)}(k-p), \tag{3.21}$$

where the factor $\frac{1}{2}$ is a symmetry factor corresponding to the exchange of the two propagators. We perform again a Wick rotation, which leads to,

$$S_{\text{eff}}[\Sigma]\big|_{D^2} = \frac{1}{2}\int \frac{(-\mathrm{i})\mathrm{d}^2k}{(2\pi)^2}\frac{(-\mathrm{i})\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{i}^2}{(-k^2-|\sigma|^2)(-p^2-|\sigma|^2)}\delta^{(2)}(k-p). \tag{3.22}$$

Integration over p and switching to polar coordinates yields the following expression,

$$S_{\text{eff}}[\Sigma]\big|_{D^2} = \frac{1}{2}\int \frac{k\,\mathrm{d}k\mathrm{d}\varphi}{(2\pi)^2}\frac{1}{(k^2+|\sigma|^2)^2}. \tag{3.23}$$

Next we integrate over $\mathrm{d}\varphi$ and substitute $u = k^2 + |\sigma|^2$, such that $\mathrm{d}u = 2k\mathrm{d}k$,

$$S_{\text{eff}}[\Sigma]\big|_{D^2} = \frac{1}{2}\int \frac{\mathrm{d}u}{4\pi}\frac{1}{u^2}. \tag{3.24}$$

Again, using an upper cut-off Λ for the momentum integration, we arrive at the result:

$$S_{\text{eff}}[\Sigma]\big|_{D^2} = -\frac{1}{8\pi}\frac{1}{u} = -\frac{1}{8\pi}\left(\frac{1}{\Lambda^2+|\sigma|^2} - \frac{1}{|\sigma|^2}\right). \tag{3.25}$$

This result can be further simplified to yield

$$S_{\text{eff}}[\Sigma]\big|_{D^2} = \frac{1}{8\pi}\frac{1}{|\sigma|^2}\frac{\Lambda^2}{\Lambda^2+|\sigma|^2}. \tag{3.26}$$

[3] That is, we treat the integral over the real k^0 line as a contour integral over the upper half plane, then rotating the contour by $\frac{\pi}{2}$ counter-clockwise. This is possible, because in the process of doing so, we do not cross any poles with the contour because of the Feynman pole prescription in the propagator. Finally, we perform a substitution $k^0 \to k^2$.

3.3 Effective Potential

If we look at the terms linear in D and v_{01} in Equation (3.4), we get

$$S_{\text{eff}}\big|_{\text{linear in } D,\, v_{01}} = -\frac{1}{4\pi}\log\left(\tfrac{\Lambda^2+|\sigma|^2}{|\sigma|^2}\right)D + \frac{\varphi_\sigma}{2\pi}v_{01}, \tag{3.27}$$

where the second term originates in the axial anomaly. For large values of the cut-off Λ, we can use $\log(1+x) \approx \log(x)$ for large x, which yields:

$$\begin{aligned} S_{\text{eff}}\big|_{\text{linear in } D,\, v_{01}} &= -\frac{1}{4\pi}\log\left(\tfrac{\Lambda^2+|\sigma|^2}{|\sigma|^2}\right)D + \frac{\varphi_\sigma}{2\pi}v_{01} && (3.28a)\\ &= -\frac{1}{4\pi}\log\left(1+\tfrac{\Lambda^2}{|\sigma|^2}\right)D + \frac{\varphi_\sigma}{2\pi}v_{01} && (3.28b)\\ &\approx -\frac{1}{4\pi}\log\left(\tfrac{\Lambda^2}{|\sigma|^2}\right)D + \frac{\varphi_\sigma}{2\pi}v_{01} && (3.28c)\\ &= -\frac{1}{2\pi}\log\left(\tfrac{\Lambda}{|\sigma|}\right)D + \frac{\varphi_\sigma}{2\pi}v_{01} && (3.28d) \end{aligned}$$

The quantum effective potential is now given as follows, cf. Equation (3.8)

$$S_{\text{eff}}[\Sigma] = \ldots - D\,\mathfrak{Re}\{\partial_\sigma \widetilde{W}_{\text{eff}}\} - v_{01}\mathfrak{Im}\{\partial_\sigma \widetilde{W}_{\text{eff}}\} + \ldots \tag{3.29}$$

This means that

$$\mathfrak{Re}\{\partial_\sigma \widetilde{W}_{\text{eff}}\} = \frac{1}{2\pi}\log\left(\tfrac{\Lambda}{|\sigma|}\right), \tag{3.30}$$

$$\mathfrak{Im}\{\partial_\sigma \widetilde{W}_{\text{eff}}\} = -\frac{\varphi_\sigma}{2\pi} \tag{3.31}$$

Finally, we can write down an equation for the effective potential:

$$\partial_\sigma \widetilde{W}_{\text{eff}} = \frac{1}{2\pi}\log\left(\tfrac{\Lambda}{|\sigma|}\right) - \mathrm{i}\frac{\varphi_\sigma}{2\pi}. \tag{3.32}$$

We can now reabsorb the phase φ_σ into its field:

$$\begin{aligned} \partial_\sigma \widetilde{W}_{\text{eff}} &= \frac{1}{\pi}\log\left(\tfrac{\Lambda}{|\sigma|}\right) - \mathrm{i}\frac{\varphi_\sigma}{\pi} && (3.33a)\\ &= \frac{1}{\pi}\log\Lambda - \frac{1}{\pi}\log|\sigma| - \frac{1}{\pi}\log \mathrm{e}^{\mathrm{i}\varphi_\sigma} && (3.33b)\\ &= \frac{1}{\pi}\log\Lambda - \frac{1}{\pi}\log|\sigma|\mathrm{e}^{\mathrm{i}\varphi_\sigma} && (3.33c)\\ &= \frac{1}{\pi}\log\Lambda - \frac{1}{\pi}\log\sigma && (3.33d)\\ &= \frac{1}{\pi}\log\left(\tfrac{\Lambda}{\sigma}\right). && (3.33e) \end{aligned}$$

This means, the quantum effective potential is given by an integration with respect to σ:

$$\begin{aligned} \widetilde{W}_{\text{eff}} &= -\frac{1}{\pi}\int \mathrm{d}\sigma \log\left(\tfrac{\sigma}{\Lambda}\right) && (3.34)\\ &= -\frac{1}{\pi}\sigma\left[\log\left(\tfrac{\sigma}{\Lambda}\right) - 1\right] && (3.35)\\ &= -\frac{1}{\pi}\sigma\left[\log\left(\tfrac{\sigma}{\Lambda}\right) - \log \mathrm{e}\right] && (3.36) \end{aligned}$$

$$= -\frac{1}{\pi}\,\sigma\log\left(\tfrac{\sigma}{\mathrm{e}\Lambda}\right) \tag{3.37}$$
$$= -\frac{1}{\pi}\,\sigma\log\left(\tfrac{\sigma}{\Lambda}\right). \tag{3.38}$$

In the last line, we redefined $\Lambda \to \frac{\Lambda}{\mathrm{e}}$.

Appendix A

Proofs

This appendix collects all calculations that were left out in the main text for better readability.

A.1 Proof of Equation (2.4)

We need to show the following anti-commutator relations:

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2\mathrm{i}\partial_\pm, \quad \{D_\pm, \overline{D}_\pm\} = 2\mathrm{i}\partial_\pm, \quad \{D_\pm, \mathcal{Q}_\pm\} = 0. \tag{A.1}$$

The SUSY generators and the fermionic covariant derivative are defined as:

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm, \qquad D_\pm = \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm, \tag{A.2}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - \mathrm{i}\theta^\pm\partial_\pm, \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\partial_\pm. \tag{A.3}$$

Since the objects in the anti-commutators are operators, we let them act on a test function f:

$$\begin{aligned}
\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\}f &= \left(\partial_{\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)\left(-\partial_{\overline{\theta}^\pm} - \mathrm{i}\theta^\pm\partial_\pm\right)f + \left(-\partial_{\overline{\theta}^\pm} - \mathrm{i}\theta^\pm\partial_\pm\right)\left(\partial_{\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)f \\
&= -\partial_{\theta^\pm}\partial_{\overline{\theta}^\pm}f - \mathrm{i}\partial_\pm f + \mathrm{i}\theta^\pm\partial_{\theta^\pm}\partial_\pm f - \mathrm{i}\overline{\theta}^\pm\partial_\pm\partial_{\overline{\theta}^\pm}f + \overline{\theta}^\pm\theta^\pm\partial_\pm\partial_\pm f \\
&\quad - \partial_{\overline{\theta}^\pm}\partial_{\theta^\pm}f - \mathrm{i}\partial_\pm f + \mathrm{i}\overline{\theta}^\pm\partial_\pm\partial_{\overline{\theta}^\pm}f - \mathrm{i}\theta^\pm\partial_{\theta^\pm}\partial_\pm f + \theta^\pm\overline{\theta}^\pm\partial_\pm\partial_\pm f \\
&= -2\mathrm{i}\partial_\pm f, \qquad\qquad (\mathrm{A}.4)\\
\{D_\pm, \overline{D}_\pm\}f &= \left(\partial_{\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)\left(-\partial_{\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\partial_\pm\right)f + \left(-\partial_{\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\partial_\pm\right)\left(\partial_{\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)f \\
&= -\partial_{\theta^\pm}\partial_{\overline{\theta}^\pm}f + \mathrm{i}\partial_\pm f - \mathrm{i}\theta^\pm\partial_{\theta^\pm}\partial_\pm f + \mathrm{i}\overline{\theta}^\pm\partial_\pm\partial_{\overline{\theta}^\pm}f + \overline{\theta}^\pm\theta^\pm\partial_\pm\partial_\pm f \\
&\quad - \partial_{\overline{\theta}^\pm}\partial_{\theta^\pm}f + \mathrm{i}\partial_\pm f - \mathrm{i}\overline{\theta}^\pm\partial_\pm\partial_{\overline{\theta}^\pm}f + \mathrm{i}\theta^\pm\partial_{\theta^\pm}\partial_\pm f + \theta^\pm\overline{\theta}^\pm\partial_\pm\partial_\pm f \\
&= 2\mathrm{i}\partial_\pm f, \qquad\qquad (\mathrm{A}.5)\\
\{D_\pm, \mathcal{Q}_\pm\}f &= \left(\partial_{\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)\left(\partial_{\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)f + \left(\partial_{\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)\left(\partial_{\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm\right)f \\
&= -\mathrm{i}\overline{\theta}^\pm\partial_{\theta^\pm}\partial_\pm f - \mathrm{i}\overline{\theta}^\pm\partial_{\theta^\pm}\partial_\pm f + \mathrm{i}\overline{\theta}^\pm\partial_{\theta^\pm}\partial_\pm f + \mathrm{i}\overline{\theta}^\pm\partial_{\theta^\pm}\partial_\pm f \qquad (\mathrm{A}.6)\\
&= 0. \qquad\qquad (\mathrm{A}.7)
\end{aligned}$$

A.2 Proof of Equation (2.8)

A chiral superfield Φ is defined via the equations $\overline{D}_\pm\Phi = 0$ with the fermionic covariant derivative

$$\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\partial_\pm. \tag{A.8}$$

This can be simplified by doing a coordinate transformation $(x, \theta, \overline{\theta}) \to (y, \theta, \overline{\theta})$ using new coordinates $y^\pm := x^\pm - \mathrm{i}\theta^\pm\overline{\theta}^\pm$:

$$\begin{aligned}
\frac{\partial}{\partial x^\pm} &= \frac{\partial y^\pm}{\partial x^\pm}\frac{\partial}{\partial y^\pm} + \frac{\partial\theta^\pm}{\partial x^\pm}\frac{\partial}{\partial\theta^\pm} + \frac{\partial\overline{\theta}^\pm}{\partial x^\pm}\frac{\partial}{\partial\overline{\theta}^\pm} = \frac{\partial}{\partial y^\pm},\\
\frac{\partial}{\partial\overline{\theta}^\pm} &= \frac{\partial y^\pm}{\partial\overline{\theta}^\pm}\frac{\partial}{\partial y^\pm} - \frac{\partial\theta^\pm}{\partial\overline{\theta}^\pm}\frac{\partial}{\partial\theta^\pm} - \frac{\partial\overline{\theta}^\pm}{\partial\overline{\theta}^\pm}\frac{\partial}{\partial\overline{\theta}^\pm} = \mathrm{i}\theta^\pm\frac{\partial}{\partial y^\pm} - \frac{\partial}{\partial\overline{\theta}^\pm}\\
\Rightarrow \overline{D}_\pm &= -\mathrm{i}\theta^\pm\frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\frac{\partial}{\partial y^\pm} = \frac{\partial}{\partial\overline{\theta}^\pm}
\end{aligned}$$

Therefore, a general chiral superfield is given by any function that depends only on $y^\pm$ and $\theta^\pm$:

$$\Phi(y^\pm, \theta^\pm) = \phi(y) + \theta^a\psi_a(y) + \theta^+\theta^-F(y). \tag{A.10}$$

A.3 Proof of Equation (2.9)

We want to show that a chiral superfield, which can be expressed as

$$\Phi(y^\pm, \theta^\pm) = \phi(y) + \theta^a\psi_a(y) + \theta^+\theta^-F(y) \tag{A.11}$$

in terms of coordinates $(y^\pm, \theta^\pm)$ is given by the following expression when switching to coordinates $(x^\pm, \theta^\pm, \overline{\theta}^\pm)$

$$\begin{aligned}
\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \phi(x^\pm) &- \mathrm{i}\theta^+\overline{\theta}^+\partial_+\phi(x^\pm) - \mathrm{i}\theta^-\overline{\theta}^-\partial_-\phi(x^\pm) - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\phi(x^\pm)\\
&+ \theta^+\psi_+(x^\pm) - \mathrm{i}\theta^+\theta^-\overline{\theta}^-\partial_-\psi_+(x^\pm) + \theta^-\psi_-(x^\pm)\\
&- \mathrm{i}\theta^-\theta^+\overline{\theta}^+\partial_+\psi_-(x^\pm) + \theta^+\theta^-F(x^\pm),
\end{aligned} \tag{A.12}$$

where $y^\pm := x^\pm - \mathrm{i}\theta^\pm\overline{\theta}^\pm$. To do this, we have to perform a Taylor expansion of y around x. This is possible since all θ coordinates square to zero. A two-dimensional Taylor expansion reads:

$$\begin{aligned}
f(x, y) = f(x_0, y_0) &+ f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y\\
&+ \frac{1}{2}\left[f_{xx}(x_0, y_0)(\Delta x)^2 + 2f_{xy}(x_0, y_0)\Delta x\Delta y + f_{yy}(x_0, y_0)(\Delta y)^2\right]\\
&+ \mathcal{O}\big((\Delta x)^3 + (\Delta y)^3\big),
\end{aligned} \tag{A.13}$$

Here, $\Delta x = -\mathrm{i}\theta^+\overline{\theta}^+$ and $\Delta y = -\mathrm{i}\theta^-\overline{\theta}^-$. We will treat the components of Φ separately. The scalar field ϕ transforms as

$$\phi(y^\pm) = \phi(x^\pm) - \mathrm{i}\theta^+\overline{\theta}^+\partial_+\phi(x^\pm) - \mathrm{i}\theta^-\overline{\theta}^-\partial_-\phi(x^\pm) - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\phi(x^\pm), \tag{A.14}$$

the spinor field ψ_+ transforms like

$$\theta^+\psi_+(y^\pm) = \theta^+\psi_+(x^\pm) - \mathrm{i}\theta^+\theta^-\overline{\theta}^-\partial_-\psi_+(x^\pm), \tag{A.15}$$

and the auxiliary field F transforms as

$$\theta^+\theta^-F(y^\pm) = \theta^+\theta^-F(x^\pm). \tag{A.16}$$

A.4 Proof of Equation (2.11) – Wess-Zumino Gauge

A vector superfield is constrained by the equation $V = V^*$. We want to express it in a certain gauge, namely the Wess-Zumino gauge. A gauge transformation of V is given by

$$V \to V' = V + \mathrm{i}\overline{A} - \mathrm{i}A,$$

where A is a chiral superfield. A chiral supermultiplet consists of three fields: a complex scalar field ϕ, a spinor field ψ and an auxiliary field F. In order to see which constituent fields of V are affected by such a gauge transformation, we write down the gauge transformation separately for every term in the θ expansion:

V	$+\mathrm{i}\overline{A}$	$-\mathrm{i}A$
a	$+\mathrm{i}\overline{\phi}_A$	$-\mathrm{i}\phi_A$
$\mathrm{i}\theta^{\pm}b_{\pm} + \mathrm{i}\overline{\theta}^{\pm}\overline{b}_{\pm}$	$-\mathrm{i}\overline{\theta}^{\pm}\overline{\psi}_{A\pm}$	$-\mathrm{i}\theta^{\pm}\psi_{A\pm}$
$\mathrm{i}\theta^+\theta^- c + \mathrm{i}\overline{\theta}^+\overline{\theta}^-\overline{c}$	$+\mathrm{i}\overline{\theta}^-\overline{\theta}^+\overline{F}_A$	$-\mathrm{i}\theta^+\theta^- F_A$
$\theta^+\overline{\theta}^- d + \theta^-\overline{\theta}^+\overline{d}$		
$\theta^+\overline{\theta}^+ e_1 + \theta^-\overline{\theta}^- e_2$	$-\theta^+\overline{\theta}^+\partial_+\overline{\phi}_A - \theta^-\overline{\theta}^-\partial_-\overline{\phi}_A$	$-\theta^+\overline{\theta}^+\partial_+\phi_A - \theta^-\overline{\theta}^-\partial_-\phi_A$
$\theta^+\theta^-\overline{\theta}^{\pm}\overline{f}_{\pm} + \overline{\theta}^+\overline{\theta}^-\theta^{\pm}f_{\pm}$	$+\overline{\theta}^+\theta^-\overline{\theta}^-\partial_-\overline{\psi}_{A+}$	$-\theta^+\theta^-\overline{\theta}^-\partial_-\psi_{A+}$
	$+\overline{\theta}^-\theta^+\overline{\theta}^+\partial_+\psi_{A-}$	$-\theta^-\theta^+\overline{\theta}^+\partial_+\psi_{A-}$
$\theta^+\theta^-\overline{\theta}^+\overline{\theta}^- g$	$-\mathrm{i}\theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\overline{\phi}_A$	$+\mathrm{i}\theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\phi_A$

with the constraints that a is a real scalar field, $b_\pm$ is a complex spinor field, c and d are complex scalar fields, $e_{1,2}$ are real scalar fields, $f_\pm$ is a complex spinor field and g is a real scalar field. To arrive at the Wess-Zumino gauge, we use a particular gauge transformation to eliminate some auxiliary fields. This is summarized in the following equations:

$$a \to a + 2\mathfrak{Im}\{\phi_A\} \overset{!}{=} 0, \tag{A.17a}$$

$$b \to b - \psi_A \overset{!}{=} 0, \tag{A.17b}$$

$$c \to c - F_A \overset{!}{=} 0, \tag{A.17c}$$

$$d \to d \overset{!}{=} -\overline{\sigma} \tag{A.17d}$$

$$e_1 \to e_1 - 2\partial_+\mathfrak{Re}\{\phi_A\} \overset{!}{=} v_0 + v_1, \tag{A.17e}$$

$$e_2 \to e_2 - 2\partial_-\mathfrak{Re}\{\phi_A\} \overset{!}{=} v_0 - v_1, \tag{A.17f}$$

$$f_\pm \to f_\pm - \partial_\pm\overline{\psi}_\mp \overset{!}{=} \mathrm{i}\lambda_\pm, \tag{A.17g}$$

$$g \to g + 2\partial_+\partial_-\mathfrak{Im}\{\phi_A\} \overset{!}{=} -D, \tag{A.17h}$$

We will use $\mathfrak{Im}\{\phi_A\}$ to set $a = 0$, ψ_A to set $b = 0$ and F_A to set $c = 0$. This still leaves $\mathfrak{Re}\{\phi_A\}$ as a degree of freedom for a "conventional" gauge transformation $v_\mu \to v_\mu - \partial_\mu\mathfrak{Re}\{\phi_A\}$. We further rename the remaining fields such that we can write the vector superfield in Wess-Zumino gauge as

$$\begin{aligned} V_{\mathrm{WZ}} = {} & \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ & + \mathrm{i}\theta^-\theta^+\overline{\theta}^a\overline{\lambda}_a + \mathrm{i}\overline{\theta}^+\overline{\theta}^-\theta^a\lambda_a + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D \end{aligned} \tag{A.18}$$

A.5 Proof of Equation (2.12)

In this section we show that an action built up by D-, F-, and twisted F-terms is in fact invariant under a SUSY transformation. Such a transformation transforms a superfield $\mathcal{S}$ like $\mathcal{S} \to \mathcal{S}' = \mathcal{S} + \delta_\epsilon \mathcal{S}$ with $\delta_\epsilon = \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \overline{\epsilon}_+ \overline{\mathcal{Q}}_- + \overline{\epsilon}_- \overline{\mathcal{Q}}_+$. The SUSY generators are given by

$$\mathcal{Q}_\pm = \frac{\partial}{\partial \theta^\pm} + \mathrm{i}\overline{\theta}^\pm \partial_\pm, \quad \overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial \overline{\theta}^\pm} - \mathrm{i}\theta^\pm \partial_\pm. \tag{A.19}$$

We will now show that the variations of such terms all vanish under a d^2x integration.

D-terms. The action for a D-term transforms in the following way:

$$\int \mathrm{d}^2x\,[\mathcal{S}]_D = \int \mathrm{d}^2x\,\mathrm{d}^4\theta\,\mathcal{S}(x,\theta,\overline{\theta}) \tag{A.20}$$

$$\to \int \mathrm{d}^2x\,\mathrm{d}^4\theta\,(\mathcal{S} + \delta_\epsilon \mathcal{S}) \tag{A.21}$$

$$= \int \mathrm{d}^2x\,[\mathcal{S}]_D + \int \mathrm{d}^2x\,\mathrm{d}^4\theta\,(\epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \overline{\epsilon}_+ \overline{\mathcal{Q}}_- + \overline{\epsilon}_- \overline{\mathcal{Q}}_+)\mathcal{S} \tag{A.22}$$

We will look at the four additional terms separately.

- The term proportional to ϵ_+ reads

$$\int \mathrm{d}^2x\,\mathrm{d}^4\theta\,\epsilon_+ \left(\frac{\partial}{\partial\theta^-} + \mathrm{i}\overline{\theta}^- \partial_-\right)\mathcal{S}$$

  This term vanishes. As for the first part, the derivative removes a θ^- from the θ-expansion of $\mathcal{S}$ and then there cannot be another θ^- left to integrate over. As for the second part, this is a total derivative with respect to x^0 and x^1, whose boundary term vanishes.

- The terms proportional to ϵ_- and $\overline{\epsilon}_\pm$ can be treated in the same way.

F-terms. The action for an F-term transforms in the following way (note that $\mathrm{d}^2\theta = \mathrm{d}\theta^- \mathrm{d}\theta^+$):

$$\int \mathrm{d}^2x\,[\Phi]_F = \int \mathrm{d}^2x\,\mathrm{d}^2\theta\,\Phi \tag{A.23}$$

$$\to \int \mathrm{d}^2x\,\mathrm{d}^2\theta\,(\Phi + \delta_\epsilon \Phi) \tag{A.24}$$

$$= \int \mathrm{d}^2x\,[\Phi]_F + \int \mathrm{d}^2x\,\mathrm{d}^2\theta\,(\epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \overline{\epsilon}_+ \overline{\mathcal{Q}}_- + \overline{\epsilon}_- \overline{\mathcal{Q}}_+)\Phi \tag{A.25}$$

We will look at the four additional terms separately.

- The term proportional to ϵ_+ reads

$$\int \mathrm{d}^2x\,\mathrm{d}^2\theta\,\epsilon_+ \left(\frac{\partial}{\partial\theta^-} + \mathrm{i}\overline{\theta}^- \partial_-\right)\Phi$$

  Here, the first part would again lead to two derivatives with respect to θ^- and the second part leads to a boundary term. Both vanish.

- The term proportional to ϵ_- vanishes for the same reason.
- For the terms proportional to $\bar{\epsilon}_\pm$, we note that $\overline{\mathcal{Q}}_\pm = \overline{D}_\pm - 2\mathrm{i}\theta^\pm\partial_\pm$. This means:

$$\int \mathrm{d}^2x\,\mathrm{d}^2\theta \sum_\pm \bar{\epsilon}_\pm \left(\overline{D}_\pm - 2\mathrm{i}\theta^\pm\partial_\pm\right)\Phi$$

  The first part vanishes because of the defining relation of a chiral superfield, $\overline{D}_\pm\Phi = 0$. The second part vanishes again because it leads to a boundary term.

Twisted F-terms. The action for an F-term transforms in the following way (note that $\widetilde{\mathrm{d}^2\theta} = \mathrm{d}\bar{\theta}^-\mathrm{d}\theta^+$):

$$\int \mathrm{d}^2x\,[U]_F = \int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\,U \tag{A.26}$$

$$\to \int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\left(\Phi + \delta_\epsilon U\right) \tag{A.27}$$

$$= \int \mathrm{d}^2x\,[\Phi]_F + \int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\,(\epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \bar{\epsilon}_+\overline{\mathcal{Q}}_- + \bar{\epsilon}_-\overline{\mathcal{Q}}_+)U \tag{A.28}$$

We will look at the four additional terms separately.

- The term proportional to ϵ_+ reads

$$\int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\,\epsilon_+\left(\frac{\partial}{\partial\theta^-} + \mathrm{i}\bar{\theta}^-\partial_-\right)U$$

  Here, the first term contains a second derivative with respect to θ^+, and the second term yields a boundary term. Both vanish.

- For the term proportional to ϵ_-, we note again that $\mathcal{Q}_- = D_- + 2\mathrm{i}\bar{\theta}^-\partial_-$

$$\int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\,\epsilon_+\left(D_- + 2\mathrm{i}\bar{\theta}^-\partial_-\right)U$$

  Thus, the first term vanishes because of the defining equations of a twisted chiral superfield $\overline{D}_+U = 0 = D_-U$, and the second term yields a boundary term.

- Again, for the term proportional to $\bar{\epsilon}_+$, we can replace the SUSY generator by $\overline{\mathcal{Q}_+} = \overline{D}_+ - 2\mathrm{i}\theta^+\partial_+$

$$\int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\,\epsilon_+\left(\overline{D}_+ - 2\mathrm{i}\theta^+\partial_+\right)U$$

  The first term fulfils the equations $\overline{D}_+U = 0 = D_-U$, and the second term yields a boundary term. Both vanish.

- The term proportional to $\bar{\epsilon}_-$ reads

$$\int \mathrm{d}^2x\,\widetilde{\mathrm{d}^2\theta}\,\epsilon_+\left(-\frac{\partial}{\partial\bar{\theta}^-} - \mathrm{i}\theta^-\partial_-\right)U$$

  Here, the first term contains a second derivative with respect to $\bar{\theta}^-$, and the second term yields a boundary term. Both vanish.

A.6 Proof of Equation (2.20)

We want to show that the superfield strength $\Sigma = \overline{D}_+ D_- V$ of the vector superfield V is a twisted chiral superfield, i.e. it fulfils the following two equations:

$$\overline{D}_+ \Sigma = 0, \qquad D_- \Sigma = 0. \tag{A.29}$$

This is easy to show because of $\overline{D}_+ \overline{D}_+ = 0$:

$$\begin{aligned} \overline{D}_+ \overline{D}_+ f &= \Big(-\partial_{\overline{\theta}^\pm} + \mathrm{i}\theta^\pm \partial_\pm\Big)\Big(-\partial_{\overline{\theta}^\pm} + \mathrm{i}\theta^\pm \partial_\pm\Big) f \\ &= \mathrm{i}\theta^+ \partial_{\overline{\theta}^\pm} \partial_+ f - \mathrm{i}\theta^+ \partial_{\overline{\theta}^\pm} \partial_+ f = 0 \end{aligned} \tag{A.30}$$

and similarly $D_- D_- = 0$. Additionaly, we need the anti-commutator $\{D_-, \overline{D}_+\}$:

$$\begin{aligned} \{D_-, \overline{D}_+\} f &= \Big(\partial_{\theta^-} - \mathrm{i}\overline{\theta}^- \partial_-\Big)\Big(-\partial_{\overline{\theta}^+} + \mathrm{i}\theta^+ \partial_+\Big) f + \Big(-\partial_{\overline{\theta}^+} + \mathrm{i}\theta^+ \partial_+\Big)\Big(\partial_{\theta^-} - \mathrm{i}\overline{\theta}^- \partial_-\Big) f \\ &= -\partial_{\theta^-}\partial_{\overline{\theta}^+} f - \mathrm{i}\theta^+ \partial_{\theta^-}\partial_+ f + \mathrm{i}\overline{\theta}^- \partial_- \partial_{\overline{\theta}^+} f + \overline{\theta}^- \theta^+ \partial_+ \partial_- f \\ &\quad - \partial_{\overline{\theta}^+}\partial_{\theta^-} f - \mathrm{i}\overline{\theta}^- \partial_- \partial_{\overline{\theta}^+} f + \mathrm{i}\theta^+ \partial_{\theta^-}\partial_+ f + \theta^+ \overline{\theta}^- \partial_+ \partial_- f \\ &= 0. \end{aligned} \tag{A.31}$$

Using the above equations we can evaluate the defining equations of a twisted chiral superfield:

$$\overline{D}_+ \Sigma = \overline{D}_+ \overline{D}_+ D_- V = 0, \tag{A.32}$$

$$D_- \Sigma = D_- \overline{D}_+ D_- V = \{D_-, \overline{D}_+\} D_- V - \overline{D}_+ D_- D_- V = 0. \tag{A.33}$$

A.7 Proof of Equation (2.21)

We want to show that the super field strength $\Sigma = \overline{D}_+ D_- V$ is invariant under gauge transformations of V, where $V \to V' = V + \mathrm{i}\overline{A} - \mathrm{i}A$. A is a chiral superfield, defined via $\overline{D}_\pm A = 0$ and for its conjugate $D_\pm \overline{A} = 0$. This is straightforward to show using the anti-commutation relation $\{D_-, \overline{D}_+\} = 0$, derived in Appendix A.6. The transformation yields:

$$\begin{aligned} \Sigma \to \Sigma' &= \overline{D}_+ D_- V + \mathrm{i}\overline{D}_+ \underbrace{D_- \overline{A}}_{0} - \mathrm{i}\overline{D}_+ D_- A \\ &= \overline{D}_+ D_- V - \mathrm{i}\underbrace{\{\overline{D}_+, D_-\}}_{0} D_- A + \mathrm{i}D_- \underbrace{\overline{D}_+ A}_{0} \\ &= \overline{D}_+ D_- V. \end{aligned} \tag{A.34}$$

A.8 Proof of Equation (2.22)

We want to show that the superfield strength Σ of the vector superfield V with components

$$\begin{aligned} V_{\mathrm{WZ}} = \theta^- \overline{\theta}^- (v_0 - v_1) + \theta^+ \overline{\theta}^+ (v_0 + v_1) - \theta^- \overline{\theta}^+ \sigma - \theta^+ \overline{\theta}^- \overline{\sigma} \\ + \mathrm{i}\theta^- \theta^+ \overline{\theta}^a \overline{\lambda}_a + \mathrm{i}\overline{\theta}^+ \overline{\theta}^- \theta^a \lambda_a + \theta^- \theta^+ \overline{\theta}^+ \overline{\theta}^- D, \end{aligned} \tag{A.35}$$

is given in terms of these compontent fields as

$$\Sigma(\tilde{y}^{\pm}) = \sigma(\tilde{y}^{\pm}) + \mathrm{i}\theta^+\overline{\lambda}_+(\tilde{y}^{\pm}) - \mathrm{i}\overline{\theta}^-\lambda_-(\tilde{y}^{\pm}) + \theta^+\overline{\theta}^-\left[D(\tilde{y}^{\pm}) - \mathrm{i}v_{01}(\tilde{y}^{\pm})\right], \tag{A.36}$$

with $\tilde{y}^{\pm} \equiv x^{\pm} \mp \mathrm{i}\theta^{\pm}\overline{\theta}^{\pm}$, $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. To show this, we calculate Σ according to its definition as twisted chiral superfield, $\Sigma = \overline{D}_+ D_- V$:

$$\begin{aligned}
\overline{D}_+ D_- V = &\left(-\partial_{\overline{\theta}^+} + \mathrm{i}\theta^+\partial_+\right)\left(\partial_{\theta^-} - \mathrm{i}\overline{\theta}^-\partial_-\right)\Big[\theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) \\
&-\theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} + \mathrm{i}\theta^-\theta^+\overline{\theta}^a\overline{\lambda}_a + \mathrm{i}\overline{\theta}^+\overline{\theta}^-\theta^a\lambda_a + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^-D\Big] \\
= &\left(-\partial_{\overline{\theta}^+} + \mathrm{i}\theta^+\partial_+\right)\Big[\overline{\theta}^-(v_0 - v_1) - \overline{\theta}^+\sigma + \mathrm{i}\theta^+\overline{\theta}^a\overline{\lambda}_a + \mathrm{i}\overline{\theta}^+\overline{\theta}^-\lambda_- \\
&+\theta^+\overline{\theta}^+\overline{\theta}^-D - \mathrm{i}\overline{\theta}^-\theta^+\overline{\theta}^+\partial_-(v_0 + v_1) + \mathrm{i}\overline{\theta}^-\theta^-\overline{\theta}^+\partial_-\sigma + \overline{\theta}^-\theta^-\theta^+\overline{\theta}^+\partial_-\overline{\lambda}_+\Big] \\
= &\,\sigma + \mathrm{i}\theta^+\overline{\lambda}_+ - \mathrm{i}\overline{\theta}^-\lambda_- + \theta^+\overline{\theta}^-D + \mathrm{i}\overline{\theta}^-\theta^+\partial_-(v_0 + v_1) - \mathrm{i}\overline{\theta}^-\theta^-\partial_-\sigma \\
&+ \overline{\theta}^-\theta^-\theta^+\partial_-\overline{\lambda}_+ + \mathrm{i}\theta^+\overline{\theta}^-\partial_+(v_0 - v_1) - \mathrm{i}\theta^+\overline{\theta}^+\partial_+\sigma - \theta^+\overline{\theta}^+\overline{\theta}^-\partial_+\lambda_- \\
&- \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+\partial_+\partial_-\sigma.
\end{aligned} \tag{A.37}$$

The last expression can be interpreted as a Taylor expansion from coordinates $\tilde{y}^{\pm} := x^{\pm} \mp \mathrm{i}\theta^{\pm}\overline{\theta}^{\pm}$ around $x^{\pm}$. The complex scalar field σ changes according to

$$\sigma(\tilde{y}^{\pm}) = \sigma(x^{\pm}) - \mathrm{i}\theta^+\overline{\theta}^+\partial_+\sigma(x^{\pm}) + \mathrm{i}\theta^-\overline{\theta}^-\partial_-\sigma(x^{\pm}) + \theta^+\overline{\theta}^+\theta^-\overline{\theta}^-\partial_+\partial_-\sigma, \tag{A.38}$$

the spinor fields ψ are multiplied to one θ coordinate, so their expansion stops earlier,

$$\theta^+\overline{\lambda}_+(\tilde{y}^{\pm}) = \theta^+\overline{\lambda}_+(x^{\pm}) + \mathrm{i}\theta^-\overline{\theta}^-\theta^+\partial_-\overline{\lambda}_+(x^{\pm}), \tag{A.39}$$

and the auxiliary field D as well as v_{01} do not change at all because of their fermionic prefactors. This way, the desired shape of Σ can be obtained via $\tilde{y}^{\pm}$ coordinates.

A.9 Proof of Equation (3.1)

With the vector superfield in Wess-Zumino gauge, Equation (2.11),

$$\begin{aligned}
V_{\mathrm{WZ}} = &\,\theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\
&+ \mathrm{i}\theta^-\theta^+\overline{\theta}^a\overline{\lambda}_a + \mathrm{i}\overline{\theta}^+\overline{\theta}^-\theta^a\lambda_a + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^-D,
\end{aligned} \tag{A.40}$$

it is straightforward to calculate higher powers of this vector field, since every quadratic theta coordinate vanishes:

$$\begin{aligned}
V^2 &= \theta^-\overline{\theta}^-\theta^+\overline{\theta}^+(v_0 - v_1)(v_0 + v_1) + \theta^+\overline{\theta}^+\theta^-\overline{\theta}^-(v_0 + v_1)(v_0 - v_1) \\
&\quad + \theta^-\overline{\theta}^+\theta^+\overline{\theta}^-\sigma\overline{\sigma} + \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+\overline{\sigma}\sigma \\
&= 2\theta^-\theta^+\overline{\theta}^+\overline{\theta}^-(v^\mu v_\mu - |\sigma|^2)
\end{aligned} \tag{A.41}$$

A.10 Proof of Equation (3.3) – Kinetic Lagrangian

The kinetic Lagrangian is given via a D-term:

$$\mathcal{L}_{\mathrm{kin}} = [\overline{\Phi}(1 + V + \tfrac{1}{2}V^2)\Phi]_D. \tag{A.42}$$

The D-terms signifies that we do not have to multiply every term, but instead only choose terms where four fermionic coordinates meet. We will visualize this in a table:

i	(0) 1	(1) $\theta^\pm$	(2) $\overline{\theta}^\pm$	(3) $\theta^\pm\overline{\theta}^\pm$	(4) $\theta^\pm\overline{\theta}^\mp$	(5) $\theta^+\theta^-$	(6) $\overline{\theta}^+\overline{\theta}^-$	(7) $\theta^+\theta^-\overline{\theta}^\pm$	(8) $\overline{\theta}^+\overline{\theta}^-\overline{\theta}^\pm$	(9) $\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$
$\overline{\Phi}$	$\overline{\phi}$		$-\overline{\psi}_\pm$	$\mathrm{i}\partial_\pm\overline{\phi}$			$-\overline{F}$		$\mp\mathrm{i}\partial_\pm\overline{\psi}_\mp$	$\partial_+\partial_-\overline{\phi}$
1										
V				$v_0 \pm v_1$	$\overline{\sigma}$ σ			$-\mathrm{i}\overline{\lambda}_\pm$	$\mathrm{i}\lambda_\pm$	$-D$
$\frac{1}{2}V^2$										$\lvert\sigma\rvert^2 - v_\mu v^\mu$
Φ	ϕ	$\psi_\pm$		$-\mathrm{i}\partial_\pm\phi$		F		$\pm\mathrm{i}\partial_\pm\psi_\pm$		$\partial_+\partial_-\phi$

TABLE A.1: Visualization of which terms to keep in the D-term.

We will now list every combination of possible products of $\overline{\Phi}$, $(1 + V + \frac{1}{2}V^2)$ and Φ, which we label by $\mathcal{L}_{ijk}$. Note that the order of the terms is important, since we integrate by $\mathrm{d}^4\theta = \mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\overline{\theta}^-\mathrm{d}\overline{\theta}^+$ from the left. This means that some additional minus signs will appear.

$$\mathcal{L}_{009} = -\overline{\phi}\partial_+\partial_-\phi \tag{A.43a}$$
$$\mathcal{L}_{207} = -\mathrm{i}\overline{\psi}_+\partial_-\psi_+ - \mathrm{i}\overline{\psi}_-\partial_+\psi_- \tag{A.43b}$$
$$\mathcal{L}_{303} = \partial_+\overline{\phi}\partial_-\phi + \partial_-\overline{\phi}\partial_+\phi \tag{A.43c}$$
$$\mathcal{L}_{801} = \mathrm{i}\partial_+\overline{\psi}_-\psi_- + \mathrm{i}\partial_-\overline{\psi}_+\psi_+ \tag{A.43d}$$
$$\mathcal{L}_{900} = -\partial_+\partial_-\overline{\phi}\phi \tag{A.43e}$$
$$\mathcal{L}_{605} = |F|^2 \tag{A.43f}$$
$$\mathcal{L}_{033} = -\mathrm{i}\overline{\phi}(v_0 + v_1)\partial_-\phi - \mathrm{i}\overline{\phi}(v_0 - v_1)\partial_+\phi \tag{A.43g}$$
$$\mathcal{L}_{231} = -\overline{\psi}_+(v_0 - v_1)\psi_+ - \overline{\psi}_-(v_0 + v_1)\psi_- \tag{A.43h}$$
$$\mathcal{L}_{330} = \mathrm{i}\partial_+\overline{\phi}(v_0 - v_1)\phi + \mathrm{i}\partial_-\overline{\phi}(v_0 + v_1)\phi \tag{A.43i}$$
$$\mathcal{L}_{241} = -\overline{\psi}_+\overline{\sigma}\psi_- - \overline{\psi}_-\sigma\psi_+ \tag{A.43j}$$
$$\mathcal{L}_{270} = \mathrm{i}\overline{\psi}_+\overline{\lambda}_-\phi - \mathrm{i}\overline{\psi}_-\overline{\lambda}_+\phi \tag{A.43k}$$
$$\mathcal{L}_{081} = \mathrm{i}\overline{\phi}\lambda_+\psi_- - \mathrm{i}\overline{\phi}\lambda_-\psi_+ \tag{A.43l}$$
$$\mathcal{L}_{090} = -|\sigma|^2\overline{\phi}\phi + \overline{\phi}v_\mu v^\mu\phi + D|\phi|^2 \tag{A.43m}$$

Using the following definitions for the covariant derivative and the $\pm$ derivatives,

$$D_\mu = \partial_\mu + \mathrm{i}v_\mu, \qquad \partial_\pm = \frac{1}{2}\left(\partial_0 \pm \partial_1\right), \tag{A.44}$$

as well as doing some integrations by part and omitting any boundary terms, we arrive at the required result:

$$\begin{aligned}\mathcal{L}_{\mathrm{kin}} = D^*_\mu\overline{\phi}D^\mu\phi &- \mathrm{i}\overline{\psi}_-(D_0 + D_1)\psi_- - \mathrm{i}\overline{\psi}_+(D_0 - D_1)\psi_+ \\ &+ D|\phi|^2 + |F|^2 - |\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- \\ &- \mathrm{i}\overline{\phi}\lambda_-\psi_+ + \mathrm{i}\overline{\phi}\lambda_+\psi_- + \mathrm{i}\overline{\psi}_+\overline{\lambda}_-\phi - \mathrm{i}\overline{\psi}_-\overline{\lambda}_+\phi\end{aligned} \tag{A.45}$$

A.11 Proof of Equation (3.6)

We want to show that

$$-\int \mathrm{d}^4\theta K_{\mathrm{eff}}(\Sigma, \overline{\Sigma}) = \partial_\sigma\partial_{\overline{\sigma}}K_{\mathrm{eff}}(\sigma, \overline{\sigma})|D - \mathrm{i}v_{01}|^2 + \dots \tag{A.46}$$

We assume that K_{eff} is a differentiable function of the twisted superfields Σ and its conjugate $\overline{\Sigma}$. In terms of a θ-expansion, Σ is given by

$$\Sigma(\tilde{y}^{\pm}) = \sigma(\tilde{y}^{\pm}) + \mathrm{i}\theta^{+}\overline{\lambda}_{+}(\tilde{y}^{\pm}) - \mathrm{i}\overline{\theta}^{-}\lambda_{-}(\tilde{y}^{\pm}) + \theta^{+}\overline{\theta}^{-}\left[D(\tilde{y}^{\pm}) - \mathrm{i}v_{01}(\tilde{y}^{\pm})\right], \tag{A.47}$$

$$\overline{\Sigma}(\tilde{y}^{\pm}) = \overline{\sigma}(\tilde{y}^{\pm}) + \mathrm{i}\overline{\theta}^{+}\lambda_{+}(\tilde{y}^{\pm}) - \mathrm{i}\theta^{-}\overline{\lambda}_{-}(\tilde{y}^{\pm}) + \theta^{-}\overline{\theta}^{+}\left[\overline{D}(\tilde{y}^{\pm}) + \mathrm{i}\overline{v}_{01}(\tilde{y}^{\pm})\right]. \tag{A.48}$$

The integration measure is defined to be $\mathrm{d}^4\theta \equiv \mathrm{d}\theta^{+}\mathrm{d}\theta^{-}\mathrm{d}\overline{\theta}^{-}\mathrm{d}\overline{\theta}^{+}$. In order to perform the integration, we expand K_{eff} in a Taylor expansion,

$$\begin{aligned} f(x,y) = {} & f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y \\ & + \frac{1}{2}\left[f_{xx}(x_0,y_0)(\Delta x)^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right] \\ & + \frac{1}{6}\Big[f_{xxx}(x_0,y_0)(\Delta x)^3 + 2f_{xxy}(x_0,y_0)(\Delta x)^2\Delta y \\ & \quad + 2f_{xyy}(x_0,y_0)\Delta x(\Delta y)^2 + f_{yyy}(x_0,y_0)(\Delta y)^3\Big] \\ & + \frac{1}{24}\left[f_{xxxx}(x_0,y_0)(\Delta x)^4 + 4f_{xxyy}(x_0,y_0)(\Delta x)^2(\Delta x)^2 + \ldots\right] \\ & + \mathcal{O}\big((\Delta x)^5 + (\Delta y)^5\big), \end{aligned} \tag{A.49}$$

with $x_0 = \sigma$ and $\Delta x = \Sigma - \sigma$. The only terms remaining after a $\mathrm{d}^4\theta$ integration are terms that contain all four fermionic coordinates. The relevant terms are:

$$\Delta x\Delta y\big|_{\theta^4} = \theta^{+}\overline{\theta}^{-}\theta^{-}\overline{\theta}^{+}\left|D - \mathrm{i}v_{01}\right|^2, \tag{A.50a}$$

$$(\Delta x)^2\Delta y\big|_{\theta^4} = \theta^{+}\overline{\lambda}_{+}\overline{\theta}^{-}\lambda_{-}\theta^{-}\overline{\theta}^{+}\left[\overline{D} + \mathrm{i}\overline{v}_{01}\right], \tag{A.50b}$$

$$\Delta x(\Delta y)^2\big|_{\theta^4} = \overline{\theta}^{+}\lambda_{+}\theta^{-}\overline{\lambda}_{-}\theta^{+}\overline{\theta}^{-}\left[D - \mathrm{i}v_{01}\right], \tag{A.50c}$$

$$(\Delta x)^2(\Delta y)^2\big|_{\theta^4} = \theta^{+}\overline{\lambda}_{+}\overline{\theta}^{-}\lambda_{-}\overline{\theta}^{+}\lambda_{+}\theta^{-}\overline{\lambda}_{-}. \tag{A.50d}$$

Finally we use $\int \mathrm{d}^4\theta\,\theta^{-}\theta^{+}\overline{\theta}^{+}\overline{\theta}^{-} = 1$ to arrive at the result:

$$\begin{aligned} -\int \mathrm{d}^4\theta K_{\text{eff}}(\Sigma,\overline{\Sigma}) = {} & \partial_\sigma\partial_{\overline{\sigma}}K_{\text{eff}}(\sigma,\overline{\sigma})\left|D - \mathrm{i}v_{01}\right|^2 - \partial_\sigma\partial_\sigma\partial_{\overline{\sigma}}K_{\text{eff}}(\sigma,\overline{\sigma})\overline{\lambda}_{+}\lambda_{-}\left[\overline{D} + \mathrm{i}\overline{v}_{01}\right] \\ & + \partial_\sigma\partial_{\overline{\sigma}}\partial_{\overline{\sigma}}K_{\text{eff}}(\sigma,\overline{\sigma})\lambda_{+}\overline{\lambda}_{-}\left[D - \mathrm{i}v_{01}\right] \\ & + \partial_\sigma\partial_\sigma\partial_{\overline{\sigma}}\partial_{\overline{\sigma}}K_{\text{eff}}(\sigma,\overline{\sigma})\overline{\lambda}_{+}\lambda_{-}\lambda_{+}\overline{\lambda}_{-}. \end{aligned} \tag{A.51}$$

A.12 Proof of Equation (3.7)

We want to show that

$$\frac{1}{2}\int \widetilde{\mathrm{d}^2\theta}\widetilde{W}_{\text{eff}}(\Sigma) + c.c. = D\,\mathfrak{Re}\{\partial_\sigma\widetilde{W}_{\text{eff}}\} + v_{01}\mathfrak{Im}\{\partial_\sigma\widetilde{W}_{\text{eff}}\}. \tag{A.52}$$

We assume that $\widetilde{W}_{\text{eff}}$ is a holomorphic function of the twisted superfield Σ, *c.c.* stands for complex conjugation. In terms of a θ-expansion, Σ is given by

$$\Sigma(\tilde{y}^{\pm}) = \sigma(\tilde{y}^{\pm}) + \mathrm{i}\theta^{+}\overline{\lambda}_{+}(\tilde{y}^{\pm}) - \mathrm{i}\overline{\theta}^{-}\lambda_{-}(\tilde{y}^{\pm}) + \theta^{+}\overline{\theta}^{-}\left[D(\tilde{y}^{\pm}) - \mathrm{i}v_{01}(\tilde{y}^{\pm})\right], \tag{A.53}$$

$$\overline{\Sigma}(\tilde{y}^{\pm}) = \overline{\sigma}(\tilde{y}^{\pm}) + \mathrm{i}\overline{\theta}^{+}\lambda_{+}(\tilde{y}^{\pm}) - \mathrm{i}\theta^{-}\overline{\lambda}_{-}(\tilde{y}^{\pm}) + \theta^{-}\overline{\theta}^{+}\left[\overline{D}(\tilde{y}^{\pm}) + \mathrm{i}\overline{v}_{01}(\tilde{y}^{\pm})\right]. \tag{A.54}$$

The integration measure is defined to be $\widetilde{\mathrm{d}^2\theta} \equiv \mathrm{d}\overline{\theta}^{-}\mathrm{d}\theta^{+}$. In order to perform the integration, we expand K_{eff} in a Taylor expansion,

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$+\frac{1}{2}\big[f_{xx}(x_0,y_0)(\Delta x)^2+2f_{xy}(x_0,y_0)\Delta x\Delta y+f_{yy}(x_0,y_0)(\Delta y)^2\big]$$
$$+\mathcal{O}\big((\Delta x)^3+(\Delta y)^3\big), \tag{A.55}$$

with $x_0=\sigma$ and $\Delta x=\Sigma-\sigma$. The only terms remaining after a $\widetilde{\mathrm{d}^2\theta}$ integration are terms that contain either $\theta^+\overline{\theta}^-$ or for the complex conjuagted term $\theta^-\overline{\theta}^+$. The relevant terms are:

$$\Delta x\big|_{\widetilde{\theta}^2}=\theta^+\overline{\theta}^-\big[D-\mathrm{i}v_{01}\big], \tag{A.56a}$$
$$(\Delta x)^2\big|_{\widetilde{\theta}^2}=\theta^+\overline{\lambda}_+\overline{\theta}^-\lambda_-, \tag{A.56b}$$
$$\Delta y\big|_{\widetilde{\theta}^2}=\theta^-\overline{\theta}^+\big[\overline{D}+\mathrm{i}\overline{v}_{01}\big], \tag{A.56c}$$
$$(\Delta y)^2\big|_{\widetilde{\theta}^2}=\overline{\theta}^+\lambda_+\theta^-\overline{\lambda}_-. \tag{A.56d}$$

Finally we use $\int\widetilde{\mathrm{d}^2\theta}\,\theta^+\overline{\theta}^-=1$ and $\int\widetilde{\mathrm{d}^2\overline{\theta}}\,\theta^-\overline{\theta}^+=1$ to arrive at the result[1]:

$$\frac{1}{2}\int\widetilde{\mathrm{d}^2\theta}\widetilde{W}_{\text{eff}}(\Sigma)+c.c.=\frac{1}{2}\partial_\sigma\widetilde{W}_{\text{eff}}(\sigma)\big[D-\mathrm{i}v_{01}\big]+\frac{1}{4}\partial_\sigma\partial_\sigma\widetilde{W}_{\text{eff}}(\sigma)\lambda_-\overline{\lambda}_+$$
$$+\frac{1}{2}\partial_{\overline{\sigma}}\overline{\widetilde{W}}_{\text{eff}}(\sigma)\big[\overline{D}+\mathrm{i}\overline{v}_{01}\big]+\frac{1}{4}\partial_{\overline{\sigma}}\partial_{\overline{\sigma}}\overline{\widetilde{W}}_{\text{eff}}(\sigma)\lambda_+\overline{\lambda}_- \tag{A.57}$$
$$=D\,\mathfrak{Re}\big\{\partial_\sigma\widetilde{W}_{\text{eff}}(\sigma)\big\}+v_{01}\,\mathfrak{Im}\big\{\partial_\sigma\widetilde{W}_{\text{eff}}(\sigma)\big\}$$
$$+\frac{1}{2}\mathfrak{Re}\big\{\partial_\sigma\partial_\sigma\widetilde{W}_{\text{eff}}(\sigma)\lambda_-\overline{\lambda}_+\big\}. \tag{A.58}$$

A.13 Proof of Equation (3.12) – The Axial Anomaly

In Section 3.2, we separated the absolute value of the field σ from its phase, which we saw as a phase rotation of the spinor component fields

$$\psi_-\to\mathrm{e}^{-\mathrm{i}\varphi_\sigma/2}\psi_- \qquad \psi_-^*\to\mathrm{e}^{\mathrm{i}\varphi_\sigma/2}\psi_-^*, \tag{A.59}$$
$$\psi_+\to\mathrm{e}^{\mathrm{i}\varphi_\sigma/2}\psi_+ \qquad \psi_+^*\to\mathrm{e}^{-\mathrm{i}\varphi_\sigma/2}\psi_+^*,$$

Please note a change of notation: in the main text, $\overline{a}$ denoted a complex conjugate, but since we want to use this to denote a Dirac conjuagate, we will use a^* for complex conjugation in this section. The fields (A.59) are the components of

$$\psi=\begin{pmatrix}\psi_-\\\psi_+\end{pmatrix},\quad \psi^*=\begin{pmatrix}\psi_-^*\\\psi_+^*\end{pmatrix},\quad \psi^\dagger=\left(\psi_-^*,\psi_+^*\right),\quad \overline{\psi}=\psi^\dagger\gamma^0=\left(\psi_+^*,\psi_-^*\right) \tag{A.60}$$

The choice for the two-dimensional gamma matrices are as follows,

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \gamma^5=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{A.61}$$

where we listed a temporal, a spatial and a chiral matrix. An axial transformation in a general quantum field theory is defined as

$$\psi\to\psi'=\mathrm{e}^{\mathrm{i}\alpha\gamma^5}\psi=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha}\psi_-\\\mathrm{e}^{-\mathrm{i}\alpha}\psi_+\end{pmatrix} \qquad \mathrm{e}^{\mathrm{i}\alpha\gamma^5}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha}&0\\0&\mathrm{e}^{-\mathrm{i}\alpha}\end{pmatrix} \tag{A.62}$$

[1]Note: D is a real scalar field, $D=\overline{D}$.

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi} e^{i\alpha\gamma^5} = \left(e^{i\alpha}\psi_+^*, e^{-i\alpha}\psi_-^*\right) \tag{A.63}$$

For $\alpha = -\varphi_\sigma/2$, this coincides with Equation (A.59).

In this section, we show the axial $U(1)$ anomaly in d Minkowski spacetime dimensions, which we later simplify to $d = 1+1$. This section is based on [5–10], which employs the Fujikawa method [5]. We consider a spinor field ψ that transforms under an axial $U(1)$ transformation as follows:

$$\psi \to \psi' = e^{i\alpha(x)\gamma^5}\psi, \quad \overline{\psi} \to \overline{\psi}' = \overline{\psi}\, e^{i\alpha(x)\gamma^5}, \tag{A.64}$$

We denote with γ^5 the chiral gamma matrix, defined as $\gamma^{\text{chir}} = i^{d/2-1}\gamma^0\gamma^1\dots\gamma^{d-1}$ for an even number of spacetime dimensions. We will only consider an even number of dimensions, since for d odd, the chiral gamma matrix is proportional to the identity. To show the axial anomaly, we only consider massless fermions in the Lagrangian. This means:

$$\mathcal{L} = \overline{\psi} i \not{D} \psi, \qquad \not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + i v_\mu), \tag{A.65}$$

where v_μ is the gauge field. The action and the partition function are given by:

$$S = \int d^d x \mathcal{L}, \qquad Z = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{iS}. \tag{A.66}$$

Since the axial transformation is merely a change of the integration variable, the partition function should stay the same. Let us inverstiagte how the terms in the partition function, that is the action and the path integral measures, change.

First, the action. If we perform the axial transformation, we get

$$\begin{aligned}
\mathcal{L} = \overline{\psi} i \not{D}\psi &\to \overline{\psi}\, e^{i\alpha\gamma^5}\, i\not{D}\, e^{i\alpha\gamma^5}\psi & \text{(A.67a)}\\
&= \overline{\psi}\, e^{i\alpha\gamma^5} i\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu\psi + \overline{\psi}\, e^{i\alpha\gamma^5} i\gamma^\mu\left(\partial_\mu e^{i\alpha\gamma^5}\right)\psi & \\
&\quad + \overline{\psi}\, e^{i\alpha\gamma^5} i\gamma^\mu i v_\mu e^{i\alpha\gamma^5}\partial_\mu\psi & \text{(A.67b)}\\
&= \overline{\psi} i\gamma^\mu(\partial_\mu + i v_\mu)\psi + \overline{\psi}\, e^{i\alpha\gamma^5} i\gamma^\mu e^{i\alpha\gamma^5}(\partial_\mu i\alpha\gamma^5)\psi & \text{(A.67c)}\\
&= \overline{\psi} i\gamma^\mu D_\mu\psi - (\partial_\mu\alpha)\overline{\psi}\gamma^\mu\gamma^5\psi & \text{(A.67d)}\\
&= \overline{\psi} i\not{D}\psi - (\partial_\mu\alpha) j_A^\mu & \text{(A.67e)}\\
&= \overline{\psi} i\not{D}\psi - \partial_\mu(\alpha j_A^\mu) + \alpha\partial_\mu j_A^\mu. & \text{(A.67f)}
\end{aligned}$$

Here we used $\exp\left(i\alpha\gamma^5\right)\gamma^\mu\exp\left(i\alpha\gamma^5\right) = \gamma^\mu$ and defined the axial-vector current as $j_A^\mu := \overline{\psi}\gamma^\mu\gamma^5\psi$. The second term in Equation (A.67f) is a total derivative, so the action changes up to a surface term as

$$S \xrightarrow{\text{chiral}} S + \int d^d x\, \alpha(x)\partial_\mu j_A^\mu. \tag{A.68}$$

Classically, the action should be invariant under a change of coordinates, which would mean that $\partial_\mu j_A^\mu = 0$, i.e. the axial current is conserved. However, this will not hold true for the quantum theory.

Second, the path integral measure. The transformation of the path integral measure $\mathcal{D}\psi$ is described by a Jacobian factor $\det J$. Since we integrate over fermionic degrees of freedom we pick up a factor $(\det J)^{-1}$ each for $\mathcal{D}\psi$ and $\mathcal{D}\overline{\psi}$, instead of $(\det J)^{+1}$ as it would be the case for bosons.
The axial transformation is a unitary transformation, which we already denoted as

$$\psi \to \psi' = U\psi, \quad U = \mathrm{e}^{\mathrm{i}\alpha\gamma^5}, \tag{A.69}$$

(note that ψ and $\overline{\psi}$ change with the same unitary transformation!) such that the path integral measure transforms as

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} \to (\det U)^{-1}\mathcal{D}\psi(\det U)^{-1}\mathcal{D}\overline{\psi} = (\det U)^{-2}\mathcal{D}\psi\mathcal{D}\overline{\psi}. \tag{A.70}$$

We use Jacobi's formula, $\det \mathrm{e}^A = \mathrm{e}^{\mathrm{Tr}\, A}$, to rewrite the determinant of U:

$$(\det U)^{-2} = (\mathrm{e}^{\mathrm{Tr}\,\mathrm{i}\alpha\gamma^5})^{-2} = \mathrm{e}^{-2\mathrm{i}\,\mathrm{Tr}\,\alpha\gamma^5}. \tag{A.71}$$

Here, the trace goes over a continuous spacetime index for $\alpha(x)$ (integration over a bra-ket matrix element) and Dirac space for γ^5 (denoted by Tr_D). This, however, is ill-defined:

$$(\det U)^{-2} = \mathrm{e}^{-2\mathrm{i}\,\mathrm{Tr}\,\alpha\gamma^5} \tag{A.72a}$$
$$= \mathrm{e}^{-2\mathrm{i}\int \mathrm{d}^d x \langle x|\alpha(\hat{x})|x\rangle\, \mathrm{Tr}_D\{\gamma^5\}} \tag{A.72b}$$
$$= \mathrm{e}^{-2\mathrm{i}\int \mathrm{d}^d x\, \alpha(x)\langle x|x\rangle\, \mathrm{Tr}_D\{\gamma^5\}} \tag{A.72c}$$
$$= \mathrm{e}^{-2\mathrm{i}\int \mathrm{d}^d x\, \alpha(x)\delta^{(d)}(x-x)\, \mathrm{Tr}_D\{\gamma^5\}} \tag{A.72d}$$
$$= \mathrm{e}^{-2\mathrm{i}\int \mathrm{d}^d x\, \alpha(x)\delta^{(d)}(0)\, \mathrm{Tr}_D\{\gamma^5\}}. \tag{A.72e}$$

In Equation (A.72e), $\delta(0)$ diverges and the trace over the chiral gamma matrix vanishes. To make sense of this, Fujikawa [5] introduced a regulator.

Fujikawa method. We introduce a gauge invariant regulating function, such that

$$\alpha(x)\gamma^5 \to \lim_{\Lambda\to\infty} \alpha(x)\gamma^5 f\left(-\frac{\not{D}^2}{\Lambda^2}\right). \tag{A.73}$$

Here, $\Lambda \to \infty$ is a cutoff, and $f(\xi)$ is the smooth regulating function. The function is normalized when its argument vanishes, so that in the limit of $\Lambda \to 0$ the function does not change anything $f(0) \to 1$. Additionally, we require that $f(\infty) \to 0$. Fujikawa originally chose $f(\xi) = \exp(-\xi^2)$. With this, we can write the Jacobian factor $(\det U)^{-2}$, as described by Equation (A.71), as:

$$(\det U)^{-2} = \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\,\mathrm{Tr}\,\alpha(x)\gamma^5 f\left(-\frac{\not{D}^2}{\Lambda^2}\right)\right) \tag{A.74}$$

We calculate the trace over spacetime indices via an integral and denote the spinor trace by Tr_D: After expressing the delta function as $\delta^{(d)}(x-y) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} 1\, \mathrm{e}^{\mathrm{i}k(x-y)}$ in momentum space, the anomaly function reads,

$$(\det U)^{-2} = \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\,\mathrm{Tr}\,\alpha(x)\gamma^5 f\left(-\tfrac{\not{D}^2}{\Lambda^2}\right)\right) \tag{A.75a}$$

$$= \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\int \mathrm{d}^d x\, \langle x|\, \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, f\left(-\tfrac{\not{D}^2}{\Lambda^2}\right) |x\rangle\right) \tag{A.75b}$$

$$= \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\int \mathrm{d}^d x \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, \langle x|\, f\left(-\tfrac{\not{D}^2}{\Lambda^2}\right) |x\rangle\right) \tag{A.75c}$$

$$= \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\int \mathrm{d}^d x \frac{\mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, \langle x|k\rangle\langle k|\, f\left(-\tfrac{\not{D}^2}{\Lambda^2}\right) |x\rangle\right) \tag{A.75d}$$

$$= \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\int \mathrm{d}^d x \frac{\mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, \langle x|k\rangle\, f\left(-\tfrac{\not{D}^2}{\Lambda^2}\right) \langle k|x\rangle\right) \tag{A.75e}$$

$$= \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\int \mathrm{d}^d x \frac{\mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5 \mathrm{e}^{\mathrm{i}kx} f\left(-\tfrac{\not{D}^2}{\Lambda^2}\right) \mathrm{e}^{-\mathrm{i}kx}\right) \tag{A.75f}$$

Next we make use of the fact that plane waves shift a differential operator:

$$\mathrm{e}^{-\mathrm{i}px} f(\partial) \mathrm{e}^{\mathrm{i}px} = f(\partial + \mathrm{i}p), \tag{A.76}$$

and rescale $k \to \Lambda k$ in order to write

$$(\det U)^{-2} = \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\int \mathrm{d}^d x \frac{\Lambda^d \mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, f\left(-\tfrac{1}{\Lambda^2}\left(\not{D} - \mathrm{i}\Lambda \not{k}\right)^2\right)\right) \tag{A.77a}$$

$$= \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\Lambda^d \int \mathrm{d}^d x \frac{\mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, f\left(-\left(\tfrac{\not{D}}{\Lambda} - \mathrm{i}\not{k}\right)^2\right)\right) \tag{A.77b}$$

We now look at the regulator function f, which we will expand in a series for small $\frac{1}{\Lambda^2}$ around $-k^2$:

$$f\left(-\left(\tfrac{\not{D}}{\Lambda} - \mathrm{i}\not{k}\right)^2\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-k^2)}{n!} \left(-\left(\tfrac{\not{D}}{\Lambda} - \mathrm{i}\not{k}\right)^2 - (-k^2)\right)^n \tag{A.78a}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(-k^2)}{n!} \left(-\tfrac{\not{D}^2}{\Lambda^2} + 2\mathrm{i}\tfrac{\not{D}\not{k}}{\Lambda}\right)^n \tag{A.78b}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(-k^2)}{n!} \left(-\tfrac{\not{D}^2}{\Lambda^2} + 2\mathrm{i}\tfrac{D_\mu k^\mu}{\Lambda}\right)^n. \tag{A.78c}$$

Since there is a trace over a γ^5 matrix, we need at least d other different γ matrices so that the trace does not yield zero. This means, we should consider terms of order $n \geq d/2$. However, every additional inverse power of Λ would make the term vanish in the limit. This constrains us to the case $n = d/2$, such that the Jacobian factor reads,

$$(\det U)^{-2} = \lim_{\Lambda\to\infty} \exp\left(-2\mathrm{i}\Lambda^d \int \mathrm{d}^d x \frac{\mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, \frac{f^{(\frac{d}{2})}(-k^2)}{(d/2)!} \left(-\tfrac{\not{D}^2}{\Lambda^2} + 2\mathrm{i}\tfrac{D_\mu k^\mu}{\Lambda}\right)^{\frac{d}{2}}\right). \tag{A.79}$$

Since there are no gamma matrices present in $D_\mu k^\mu$, these terms will vanish in the trace:

$$(\det U)^{-2} = \lim_{\Lambda\to\infty} \exp\left(-\frac{2\mathrm{i}\Lambda^d}{(d/2)!} \int \mathrm{d}^d x \frac{\mathrm{d}^d k}{(2\pi)^d} \alpha(x)\, \mathrm{Tr}_D\, \gamma^5\, f^{(\frac{d}{2})}(-k^2) \left(-\tfrac{\not{D}^2}{\Lambda^2}\right)^{\frac{d}{2}}\right)$$

$$= \exp\left(-\frac{2\mathrm{i}(-1)^{\frac{d}{2}}}{(d/2)!}\int \mathrm{d}^d x\,\alpha(x)\underbrace{\frac{\mathrm{d}^d k}{(2\pi)^d}f^{(\frac{d}{2})}(-k^2)}_{\text{(ii)}}\underbrace{\mathrm{Tr}_D\,\gamma^5 \not{D}^d}_{\text{(i)}}\right). \tag{A.80}$$

We now have to consider two things: the trace in Dirac space (i) and the momentum integraion (ii). We start with the trace.

Evaluating the trace. This involves a trace over the following gamma matrices plus one chiral gamma matrix: $\mathrm{Tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_d}\}$. It is proportional to the d-dimensional epsilon tensor with proportionality factor $\mathrm{i}^{1-d/2}2^{d/2}$:

$$\mathrm{Tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_d}\} = x\,\epsilon^{\mu_1\cdots\mu_d} \tag{A.81a}$$
$$\epsilon_{\mu_1\dots\mu_d}\,\mathrm{Tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_d}\} = x\,\underbrace{\epsilon^{\mu_1\cdots\mu_d}\epsilon_{\mu_1\dots\mu_d}}_{d!} \tag{A.81b}$$
$$d!\mathrm{i}^{1-d/2}\,\mathrm{Tr}\{\gamma^5\gamma^5\} = d!x \tag{A.81c}$$
$$\mathrm{i}^{1-d/2}\,\mathrm{Tr}\{\mathbb{1}\} = x \tag{A.81d}$$
$$\mathrm{i}^{1-d/2}2^{\lfloor d/2\rfloor} = x \tag{A.81e}$$
$$\mathrm{i}^{1-d/2}2^{d/2} = x \tag{A.81f}$$

where we used the definition of a generalized chiral gamma matrix for even d dimensions,

$$\gamma^5 = \frac{\mathrm{i}^{d/2-1}}{d!}\epsilon_{\mu_1\dots\mu_d}\gamma^{\mu_1}\dots\gamma^{\mu_d}, \tag{A.82}$$

and assumed d even. Thus,

$$\mathrm{Tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_d}\} = \mathrm{i}^{1-d/2}2^{d/2}\epsilon^{\mu_1\cdots\mu_d}. \tag{A.83}$$

Now we can calculate $\mathrm{Tr}\,\gamma^5\not{D}^d$:

$$\mathrm{Tr}\,\gamma^5\not{D}^d = \mathrm{Tr}\{\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_d}\}D_{\mu_1}\dots D_{\mu_d} \tag{A.84a}$$
$$= \mathrm{i}^{1-d/2}2^{d/2}\epsilon^{\mu_1\cdots\mu_d}D_{\mu_1}\dots D_{\mu_d} \tag{A.84b}$$
$$= \mathrm{i}^{1-d/2}2^{d/2}\frac{1}{2^{d/2}}\epsilon^{\mu_1\cdots\mu_d}[D_{\mu_1},D_{\mu_2}][D_{\mu_3},D_{\mu_4}]\dots[D_{\mu_{d-1}},D_{\mu_d}] \tag{A.84c}$$
$$= \mathrm{i}^{1-d/2}\mathrm{i}^{d/2}\epsilon^{\mu_1\cdots\mu_d}v_{\mu_1\mu_2}\dots v_{\mu_{d-1}\mu_d} \tag{A.84d}$$
$$= \mathrm{i}\epsilon^{\mu_1\cdots\mu_d}v_{\mu_1\mu_2}\dots v_{\mu_{d-1}\mu_d}. \tag{A.84e}$$

Here we we able to express pairs of covariant derivatives as their antisymmetric combination because of the total antisymmetric epsilon tensor and then used an expression for the commutator of two covariant derivatives, which yields

$$D_{[\mu}D_{\nu]}f = (\partial_\mu + \mathrm{i}v_\mu)(\partial_\nu + \mathrm{i}v_\nu)f - (\partial_\nu + \mathrm{i}v_\nu)(\partial_\mu + \mathrm{i}v_\mu)f \tag{A.85}$$
$$= \partial_\mu\partial_\nu f + \mathrm{i}v_\mu\partial_\nu f + \mathrm{i}(\partial_\mu v_\nu)f + \mathrm{i}v_\nu\partial_\mu f$$
$$- \partial_\mu\partial_\nu f - \mathrm{i}v_\nu\partial_\mu f - \mathrm{i}(\partial_\nu v_\mu)f - \mathrm{i}v_\mu\partial_\nu f \tag{A.86}$$
$$= \mathrm{i}(\partial_\mu v_\nu)f - \mathrm{i}(\partial_\nu v_\mu)f \tag{A.87}$$
$$= \mathrm{i}v_{\mu\nu}f. \tag{A.88}$$

The momentum integration. We have to calculate the integral

$$\text{Int.} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} f^{(\frac{d}{2})}(-k^2). \tag{A.89}$$

Let us switch to a Euclidean metric, i.e. we introduce $k_{\mathrm{E}}^{d} := \mathrm{i}k^0$, such that the integral transforms like

$$\mathrm{d}^d k \to -\mathrm{i}\,\mathrm{d}^d k_{\mathrm{E}}, \qquad k^2 \to -k_{\mathrm{E}}^2, \tag{A.90}$$

and we get

$$\text{Int.} = -\mathrm{i} \int \frac{\mathrm{d}^d k_{\mathrm{E}}}{(2\pi)^d} f^{(\frac{d}{2})}(k_{\mathrm{E}}^2). \tag{A.91}$$

We change to hyperspherical coordinates,

$$\mathrm{d}^d k_{\mathrm{E}} = S_{d-1} \mathrm{d}k_{\mathrm{E}}, \quad S_n(R) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} R^n \to S_{d-1}(k_{\mathrm{E}}) = \frac{2\pi^{d/2}}{(d/2-1)!} k_{\mathrm{E}}^{d-1},$$

so the integral becomes

$$\text{Int.} = -\mathrm{i} \frac{2\pi^{d/2}}{(d/2-1)!} \int_0^\infty \frac{\mathrm{d}k_{\mathrm{E}}}{(2\pi)^d} k_{\mathrm{E}}^{d-1} f^{(\frac{d}{2})}(k_{\mathrm{E}}^2). \tag{A.92}$$

We substitute $u = k_{\mathrm{E}}^2$ with $\mathrm{d}u = 2k_{\mathrm{E}}\mathrm{d}k_{\mathrm{E}} = 2u^{1/2}\mathrm{d}k_{\mathrm{E}}$:

$$\text{Int.} = -\mathrm{i} \frac{2\pi^{d/2}}{(d/2-1)!} \frac{1}{2} \int_0^\infty \frac{\mathrm{d}u}{(2\pi)^d} u^{-1/2} u^{d/2-1/2} f^{(\frac{d}{2})}(u) \tag{A.93a}$$

$$= -\mathrm{i} \frac{\pi^{d/2}}{(2\pi)^d (d/2-1)!} \int_0^\infty \mathrm{d}u\, u^{d/2-1} f^{(\frac{d}{2})}(u) \tag{A.93b}$$

Next we integrate $(d/2) - 1$ times by parts, so that we get

$$\text{Int.} = -\mathrm{i} \frac{\pi^{d/2}}{(2\pi)^d (d/2-1)!} (-1)^{d/2-1} (d/2-1)! \int_0^\infty \mathrm{d}u\, f'(u) \tag{A.94a}$$

$$= \mathrm{i}(-1)^{d/2} \frac{\pi^{d/2}}{(2\pi)^d} [f(\infty) - f(0)] \tag{A.94b}$$

$$= \mathrm{i}(-1)^{d/2+1} \frac{\pi^{d/2}}{(2\pi)^d} f(0). \tag{A.94c}$$

This was possible since we required the regulating function to vanish smoothly at infinity, so every intermediate boundary terms vanished. Originally, the regulating function was introduced in the limit $\Lambda \to \infty$, where it would approach 1, cf. (A.73). This means we can take $f(0) = 1$.

The result. Remembering Equation (A.80), the Jacobian factor is given by

$$(\det U)^{-2} = \exp\left(-\frac{2\mathrm{i}(-1)^{\frac{d}{2}}}{(d/2)!} \int \mathrm{d}^d x\, \alpha(x) \underbrace{\frac{\mathrm{d}^d k}{(2\pi)^d} f^{(\frac{d}{2})}(-k^2)}_{\text{(ii)}} \underbrace{\mathrm{Tr}_D\, \gamma^5 \not{D}^d}_{\text{(i)}} \right). \tag{A.95}$$

Using our results for the trace, Equation (A.84e), and the momentum integration, Equation (A.94c), we get

$$(\det U)^{-2} = \exp\left(\mathrm{i}\int \mathrm{d}^d x\, \alpha(x) \frac{(-2)}{(d/2)!(4\pi)^{\frac{d}{2}}} \epsilon^{\mu_1 \dots \mu_d} v_{\mu_1\mu_2} \dots v_{\mu_{d-1}\mu_d}\right). \tag{A.96}$$

For two spacetime dimensions, $d = 2$, this yields

$$(\det U)^{-2}\big|_{d=2} = \exp\left(\mathrm{i}\int \mathrm{d}^2 x\, \alpha(x) \frac{(-1)}{2\pi} \epsilon^{\mu\nu} v_{\mu\nu}\right) \tag{A.97}$$

which means, that for an axial transformation with $\alpha = -\frac{\varphi_\sigma}{2}$, the Lagrangian changes by $+\frac{\varphi_\sigma}{2\pi} v_{01}$.

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