Quark Content of the Neutral Pion

Stephan Hübsch*

April 2019

Abstract

We review group representations and explain why the quark content of the neutral pion is $|\pi^0\rangle = (|\bar{u}u\rangle - |\bar{d}d\rangle)/\sqrt{2}$, whereas the usual triplet state, known from quantum mechanics, is given by $|1,0\rangle = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$. The first section is mainly based on chapter 70 of Ref. [1].

1 Group Representations

We will consider a compact, non-Abelian group, for instance $SU(2)$ or $SU(3)$ which are commonly used in physics. The commutator of two generators t^a can be calculated as a linear combination of other generators, thereby defining the group's structure constants,

$$[\mathsf{t}^a, \mathsf{t}^b] = \mathrm{i} f^{abc} \mathsf{t}^c. \tag{1}$$

A representation of this group is given by a set of $D(R) \times D(R)$-matrices T^a_{R}, which follow the same commutation relation,

$$[T^a_{\mathrm{R}}, T^b_{\mathrm{R}}] = \mathrm{i} f^{abc} T^c_{\mathrm{R}} \tag{2}$$

Here, $D(R)$ is called the dimension of the representation R. If we take the complex conjugate of eq. (2), we get

$$[(T^a_{\mathrm{R}})^*, (T^b_{\mathrm{R}})^*] = -\mathrm{i} f^{abc} (T^c_{\mathrm{R}})^* \tag{3}$$

which seems like it does not obey the original commutation relation. However, we can define the complex conjugate representation $\bar{R}$ as

$$T^a_{\bar{\mathrm{R}}} = -(T^a_{\mathrm{R}})^* \tag{4}$$

and now eq. (3) looks fine again, because on the left-hand side the minus signs cancel and on the right-hand side, the minus sign is absorbed into the conjugate representation:

$$[T^a_{\bar{\mathrm{R}}}, T^b_{\bar{\mathrm{R}}}] = \mathrm{i} f^{abc} T^c_{\bar{\mathrm{R}}} \tag{5}$$

Depending on the relation between the representation R and its complex conjugate representation $\bar{R}$, we can have three different cases.

- If $T^a_{\bar{\mathrm{R}}} = T^a_{\mathrm{R}}$, the representation R is real.
- If we can find a unitary transformation V such that $T^a_{\bar{\mathrm{R}}} = V^{-1} T^a_{\mathrm{R}} V$, the representation R is pseudoreal.
- Else, the representation R is complex.

*huebsch.s.aa@m.titech.ac.jp

2 Some Representations of $SU(2)$

In order to investigate the pion, it is sufficient to only consider two quark flavors, u and d. We demand that the quarks transform under the fundamental representation of $SU(2)$, which is called $R = \mathbf{2}$. This means they can be combined into a multipled q, such that

$$q \to q' = Uq, \quad q = \begin{pmatrix} u \\ d \end{pmatrix} \tag{6}$$

where U is a 2×2-matrix, belonging to the fundamental representation of $SU(2)$. The pion consists of a quark and an antiquark, therefore we also need to know the representation for the antiquarks. This is the complex conjugated fundamental representation, also known as the antifundamental representation,

$$\bar{q} \to \bar{q}' = U_{\bar{\mathrm{R}}}\bar{q}, \quad \bar{q} = q_{\bar{\mathrm{R}}} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \tag{7}$$

The fundamental representation of $SU(2)$ is pseudoreal. This means, we can find a similarity transformation between T^a_{R} and $T^a_{\bar{\mathrm{R}}}$, where the transformation V turns out to be the second Pauli matrix,

$$T^a_{\mathbf{2}} = \tfrac{\sigma^a}{2}, \qquad V^{-1}\tfrac{\sigma^a}{2}V = -\left(\tfrac{\sigma^a}{2}\right)^*, \quad V = -\mathrm{i}\sigma^2, \det V = 1 \tag{8}$$

We include the imaginary unit in $V = \mathrm{i}\sigma^2$, such that the determinant of V is plus one and the minus sign for later convenience. If we use this knowledge in the transformation behaviour of the antiquarks, we get an interesting relation:

$$\bar{q}' = U_{\bar{\mathrm{R}}}\bar{q} = [V^{-1}UV]\bar{q} \quad \leftrightarrow \quad V\bar{q}' = U(V\bar{q}) \tag{9}$$

This equation tells us that the combination $V\bar{q}$ transforms just like q, that is, in the fundamental representation!

3 Pions as Antiquark-Quark Systems

The pions consist of a quark and an antiquark. If we combine those representations, we get $\mathbf{2}\otimes\bar{\mathbf{2}} = \mathbf{1}\oplus\mathbf{3}$. In quantum mechanics, the usual result for the coupling of two spin-1/2 particles is the singlet and the triplet state:

$$|0,0\rangle = \frac{1}{\sqrt{2}}\Big[\,|{\uparrow}\rangle_1|{\downarrow}\rangle_2 - |{\downarrow}\rangle_1|{\uparrow}\rangle_2\Big], \qquad |1,m_s\rangle = \begin{cases} |1,1\rangle = |{\uparrow}\rangle_1|{\uparrow}\rangle_2 \\ |1,0\rangle = \frac{1}{\sqrt{2}}\Big[\,|{\uparrow}\rangle_1|{\downarrow}\rangle_2 + |{\downarrow}\rangle_1|{\uparrow}\rangle_2\Big] \\ |1,-1\rangle = |{\downarrow}\rangle_1|{\downarrow}\rangle_2 \end{cases} \tag{10}$$

Here, the two states we coupled were given in the same representation. But with our quark and antiquark system, they are in different representation. Therefore we need the result of section 2, which tells us how to represent the antiquarks in the fundamental representation of the quarks:

$$V\bar{q} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} \bar{u} \\ -\bar{d} \end{pmatrix} \quad \to \quad \text{transforms under } \mathbf{2} \text{ representation!} \tag{11}$$

If we now identify the first particle with the antiquark and the second particle with the quark, we get the correspondence

$$|{\uparrow}\rangle_1 \sim \bar{u}, \quad |{\downarrow}\rangle_1 \sim -\bar{d}, \quad |{\uparrow}\rangle_2 \sim u, \quad |{\downarrow}\rangle_2 \sim d \tag{12}$$

Therefore the triplet states after the coupling of an antiquark to a quark are given by

$$|1,1\rangle = \left|\pi^+\right\rangle = |\bar{u}u\rangle \tag{13}$$

$$|1,0\rangle = \left|\pi^0\right\rangle = \frac{1}{\sqrt{2}}\Big[\,|\bar{u}u\rangle - \left|\bar{d}d\right\rangle\Big] \tag{14}$$

$$|1,-1\rangle = \left|\pi^-\right\rangle = -\left|\bar{d}d\right\rangle \overset{\text{choose phase}}{\rightarrow} \left|\bar{d}d\right\rangle \tag{15}$$

To summarize, the negative sign inside the neutral pion is a result of the fact that we first had to modify the antiquark vector to be in the same representation as the quark vector. This lead to a minus sign in front of $\bar{d}$[1], and only after doing so, we are allowed to use the usual coupling rules from quantum mechanics.

References

[1] M. Srednicki, *Quantum Field Theory,* Cambridge University Press, 2010.

[1]Remember, we set $V = -\mathrm{i}\sigma^2$. If we did not use the minus sign, then $\bar{u}$ would have gotten the minus sign, which would lead to the same relative minus sign between $\bar{u}u$ and $\bar{d}d$ due to the free choice of the phase, but doing it like this directly leads to the desired result.